Highest
Common Factor

The **highest common factor** (H.C.F.) or **highest commo** (H.C.D) of two or more integral algebraic expression integral expression of the highest degree which will divide each of them.

Consider the expressions $27a^2b^3c$, $15a^3b^5c^4$. 3 is the the numerical coefficients 27 and 15.

The highest power of a which will divide both expressions is a^2.

.... b .... b^3.

.... c .... c.

$\therefore$ the H.C.F. of the two expressions is $3a^2b^3c$.

Example. Find the H.C.F. of $15a^5b^4c^6$, $60a^3b^5$, $25a^4b^2c^2$.

The H.C.F. of 15, 60, 25 is 5.

The highest power of a which divides all the expressions is a^3.

... . b ... . b^2.

No power of c divides all three expressions.

$\therefore$ the reqd. H.C.F. $= 5a^3b^2$.

Examples. XIX. a.

Find the highest common factor of:

1. $5a^2b$, $10ab^2$.
2. x^2y^3, x^3y^2.
3. abc, $3a^2b$.
4. $6xy^2z$, $8x^3yz^2$.
5. $9a^2b^2c^2$, $15a^3bc^4$.
6. $9a^2x^4$, $21b^2x^3$.
7. $6x^2y$, $3xy^2$, $9x^2y^2$.
8. x^2y, y^2z, xy^2.
9. $3a^2c^6$, $27a^4c^4$, $18a^3c^2$.
10. $26x^3y^2$, $13x^2z^3$, $39x^2y^2z^2$
11. $35a^6b^4c^2d^3$, $20a^5c^3d^4$, $45a^3b^2d$, $10a^7b^4cd^7$.
12. $3r^2c^3$ $5a^2bc$, $7abc^2$, $9abcd$.

...und *expressions* the H.C.F. can be determined by ...n as the expressions are resolved into their

... the H.C.F. of

$a^2bx + ab^2x$ and $a^2b - b^3$.

$a + b)(a - b)$.

By inspection the

Example 2. Find

$x^2 + 2x - 80$.

$x^2 + 7x - 60 = (x + 12)(x - 5)$.

$\therefore$ the reqd. H.C.F. is $x - 5$.

Example 3. Find the H.C.F. of x^2-4, x^2+3x+2, x^2+x-2.

$$x^2-4=(x-2)(x+2),$$
$$x^2+3x+2=(x+1)(x+2),$$
$$x^2+x-2=(x-1)(x+2).$$

$\therefore$ $x+2$ is the H.C.F. reqd.

Example 4. Find the H.C.F. of $x^3-ax^2+a^2x-a^3$ and $x^3-ax^2-a^2x+a^3$.

$$x^3-ax^2+a^2x-a^3=x^2(x-a)+a^2(x-a)=(x-a)(x^2+a^2),$$
$$x^3-ax^2-a^2x+a^3=x^2(x-a)-a^2(x-a)=(x-a)(x^2-a^2)$$
$$=(x+a)(x-a)^2.$$

$\therefore$ the reqd. H.C.F. is $x-a$.

Examples. XIX. b.

Find the H.C.F. of :

1. a^2-ax, a^2+ax.
2. $5x-10$, $4x-8$.
3. x^2+xy, $xy+y^2$.
4. x^2-4, $3x-6$.
5. a^2+2ab, $ab+2b^2$.
6. x^2+xy, x^2-y^2.
7. x^2-2xy, x^2-4y^2.
8. $x^2+2xy+y^2$, x^2-y^2.
9. x^3-3ax^2, $2x^2-6ax$.
10. $15x-45$, $3x^2-27$.
11. $3x^2+12xy$, $4x^2-64y^2$.
12. $4x^2-8xy$, $3xy^2-6y^3$.
13. x^2+3x+2, x^2+6x+5.
14. $1-2x+x^2$, $1-x^2$.
15. $1+2x+x^2$, $4x-4x^3$.
16. $x^2-7x+12$, $x^2-8x+15$.
17. x^3+y^3, $5x^2-5y^2$.
18. x^2-x-20, x^2+3x-4.
19. x^2-121, $x^2+12x+11$.
20. $x^2+17x+60$, $x^2-7x-60$.
21. $3x^3+3a^3$, $2x^2+4ax+2a^2$.
22. a^3+b^3, $a^2b-ab^2+b^3$.
23. x^2+x-42, $x^2-9x+18$.
24. $4x^2+12x-72$, $3x^2-3x-18$.
25. $24a^5b^2(a+b)^2$, $21a^3b^4(a^3+b^3)$.
26. $12x^2-x-1$, $6x^2-5x+1$.
27. $2x^2+5x-3$, $7x^2-63$.
28. x^3-2x^2-x+2, x^3-x^2-4x+4.
29. $(b+c)^2-a^2$, $(c+a)^2-b^2$, $(a+b)^2-c^2$.
30. $10x^2+13x-3$, $5x^2-11x+2$, $5x^2-16x+3$.
31. $x^2-7x+10$, x^2+2x-8, $3x^2-3x-6$.
32. $(a-b)^2-c^2$, $(a+c)^2-b^2$, $(c-b)^2-a^2$.
33. $x^2-10x+25$, x^2-25, x^3-125.
34. $x^2-(a-c)x-ac$, $x^2-(a+c)x+ac$.
35. $2x^2+x-1$, $2x^2-5x+2$, $6x^2+x-2$.
36. $16x^4+36x^2+81$, $8x^3+27$.
37. x^3-x^2-3x+3, x^3-3x^2+2.
38. x^4-x^2-2x+2, $2x^3-x-1$.
39. $15x^3-19x^2+4$, $9x^3-9x^2-4x+4$.
40. $x^2-7x+10$, $4x^3-25x^2+20x+25$.

***114.** When compound expressions cannot readily be factorized we find their H.C.F. by a method analogous to the Arithmetical method.

Before attempting any such, the student must grasp the principle underlying the Arithmetical method.

Let us find the H.C.F. of 782 and 5451.

$$\begin{array}{l}
782\,)\,5451\,(\,6 \\
\quad\;\; 4692 \\
\qquad 759\,)\,782\,(\,1 \\
\qquad\qquad\;\; 759 \\
\qquad\qquad\;\; 23\,)\,759\,(\,33 \\
\qquad\qquad\qquad\quad\;\; 69 \\
\qquad\qquad\qquad\quad\;\;\; 69 \\
\qquad\qquad\qquad\quad\;\;\; \underline{69}
\end{array}$$

23 is the reqd. H.C.F.

This method depends upon the fact that if any two numbers have a common factor, the remainder, when one is divided by the other, has the same factor.

Thus in the above,

any factor common to 782 and 5451 is a factor of 759.

.... 759 and 782 23

This principle, a rigid proof of which will be given later, being true for Arithmetical numbers must also be true in Algebra, since the symbols stand for numbers.

Let us now apply it to some examples

Example 1. Find the H.C.F. of x^3+6x^2-8x-7 and $x^3+8x^2+10x+21$.

$$\begin{array}{l}
x^3+6x^2-8x-7\,)\,x^3+8x^2+10x+21\,(\,1 \\
\qquad\qquad\qquad\;\; x^3+6x^2-\;\,8x-\;\,7 \\
\qquad\qquad (a)\quad 2\,)\,2x^2+18x+28\,)\,x^3+6x^2-\;\,8x-\;\,7\,(\,x-3 \\
\qquad\qquad\qquad\quad\;\; x^2+\;\,9x+14\quad x^3+9x^2+14x \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -3x^2-22x-\;\,7 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad -3x^2-27x-42 \\
\qquad\qquad\qquad\qquad\qquad\qquad\quad (b)\quad 5\,)\,5x+35\,(\,x^2+9x+14\,)\,x+2 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad x+\;\,7\quad x^2+7x \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 2x+14 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \underline{2x+14}
\end{array}$$

$x+7$ is the reqd. H.C.F

(*a*) Here we see that 2 is a factor of $2x^2+18x+28$, *but not a factor of* x^3+6x^2-8x-7: we therefore reject it.

(*b*) We see that 5 is a factor of $5x+35$, *but not a factor of* $x^2+9x+14$: we therefore reject it.

The work will be considerably simplified if factors not common to both divisor and dividend are rejected in this way.

Time will be saved if the work is arranged as below:

$$
\begin{array}{r|r|r|l}
x & x^3+6x^2-\ 8x-\ 7 & x^3+8x^2+10x+21 & 1 \\
 & x^3+9x^2+14 & x^3+6x^2-\ 8x-\ 7 & \\
-3 & -3x^2-22x-\ 7 & 2)\,2x^2+18x+28 & \\
 & -3x^2-27x-42 & x^2+\ 9x+14 & x \quad \ldots \quad \ldots \quad (c) \\
 & 5)\,5x+35 & x^2+\ 7x & \\
 & x+\ 7 & 2x+14 & 2 \\
 & & 2x+14 &
\end{array}
$$

At the stage (c) we might have shortened the work thus. The factors of $x^2+9x+14$ are $x+2$ and $x+7$. $x+2$ is evidently not a divisor of the given expressions.

Dividing x^3+6x^2-8x-7 by $x+7$ we find that $x+7$ is the H.C.F.

When the given expressions have factors common to every term, these should be removed first, remembering that they themselves may have a common factor.

Example 2. Find the H.C.F. of

$$36x^4-78x^3+18x^2+12x \text{ and } 90x^4-207x^3+63x^2+36x.$$

$$36x^4-78x^3+18x^2+12x=6x(6x^3-13x^2+3x+2).$$

$$90x^4-207x^3+63x^2+36x=9x(10x^3-23x^2+7x+4).$$

$3x$ is the H.C.F. of $6x$ and $9x$.

We now proceed to find the H.C.F. of the remaining factors.

$$
\begin{array}{r|r|r}
3x & 6x^3-13x^2+3x+2 & 10x^3-23x^2+7x+4 \\
 & 6x^3-\ 9x^2\ \ 3x & 12x^3-26x^2+6x+4 \\
 & -\ 4x^2+6x+2 & -x)\,-2x^3+\ 3x^2+\ x \\
 & -\ 4x^2+6x+2 & 2x^2-\ 3x\ -\ 1
\end{array}
$$

$\therefore$ the reqd. H.C.F. is $3x(2x^2-3x-1)$.

Example 3. Find the H.C.F. of

$$6x^3-19x^2+11x^2+6 \text{ and } 10x^3-19x^2+2x+6$$

$$
\begin{array}{lr|r|r|ll}
 & & 6x^3-19x^2+11x+\ 6 & 10x^3-19x^2+\ \ 2x+6 & 1 & \\
(c) & & 2 & 6x^3-19x^2+11x+6 & & (a) \\
 & 3x & 12x^3-38x^2+22x+12 & x)\,4x^3\qquad\ -\ \ 9x & & (d) \\
 & & 12x^3\qquad -27x & 4x^2\qquad -\ \ 9 & & \\
 & -9 & -38x^2+49x+12 & 4x^2-98x\ +138 & 2 & \\
(b) & & -36x^2\qquad +81 & 49)\,98x\ -147 & & \\
 & & -1)\,-\ 2x^2+49x-69 & 2x\ -\ \ 3 & & \\
 & & 2x^2-49x+69 & & & \\
 & x & 2x^2-\ 3x & & & \\
 & +23 & -46x+69 & & & \\
 & & -46x+69 & & &
\end{array}
$$

The reqd. H.C.F. is $2x-3$.

N.B.—It is not necessary that the first term of the divisor should go an exact number of times into the first term of the dividend. See (*a*) and (*b*).

It is, however, sometimes convenient, as at (*c*), to introduce a factor.

At (*d*) we reject the factor x, which is not a factor of either of the given expressions.

***115.** If A and B represent any integral algebraical expression, then if A and B have a common factor, their sum or difference has the same factor.

Let p be the common factor of A and B, and C and D the quotients when we divide them by p.

$$\text{Then}\quad A = pC, \quad\text{and}\quad B = pD.$$

$$\therefore\ A + B = p(C + D),\ \textit{i.e.}\ p \text{ is a factor of } A + B.$$

In the same way $\quad A - B = p(C - D), \ \therefore\ p$ .... $A - B$.

Further if A and B have a common factor p, p is also a factor of $mA + nB$ and $mA - nB$, where mA and nB are any multiples of A and B.

Let C and D be the quotients when we divide A and B by p, so that

$$A = pC, \quad\text{and}\quad B = pD$$

$$\therefore\ mA + nB = mpC + npD$$
$$= p(mC + nD);$$

$$\therefore\ p \text{ is a factor of } mA + nB.$$

In the same way, $\quad mA - nB = p(mC - nD);$

$$\therefore\ p \text{ is a factor of } mA - nB.$$

This can often be employed to shorten the work of finding a H.C.F.

Find the H.C.F. of

$$5x^3 + 16x^2 + 23x - 5148 \quad\text{and}\quad 3x^3 + 48x^2 - 103x - 5148.$$

The difference of the two expressions

$$= 2x^3 - 32x^2 + 126x$$
$$= 2x(x^2 - 16x + 63)$$
$$= 2x(x - 7)(x - 9).$$

Now $2x$ is not a common factor, nor is $x - 7$, for 7 will not divide exactly into 5148.

$\therefore\ x - 9$ must be the H.C.F. if there is one.

*Examples. XIX. c.

Find the highest common factor of

1. $30a^2x^4 - 5a^3x^3 + 5a^5x, \quad 9ax^5 - a^3x + 2a^4.$
2. $x^4 - 2x^3y - 2x^2y^2 - 3xy^3, \quad 3x^3y + 2x^2y^2 + 2xy^3 - y^4.$

Find the highest common factor of:

3. $2x^4 - x^3 - x^2 - x - 3$, $2x^4 - 5x^3 + x^2 + 5x - 3$.
4. $2x^3 - 7x^2 + 8x - 4$, $6x^3 - 6x^2 - 11x - 2$.
5. $2x^3 - 5x + 6$, $4x^3 + x^2 - 12x + 4$.
6. $3x^3 + 14x^2 + 12x + 16$, $2x^4 + 7x^3 - 4x^2 - x - 4$.
7. $2x^4 + 9x^3 + 14x + 3$, $3x^4 + 15x^3 + 5x^2 + 10x + 2$.
8. $12x^3 + 9x^2 - 4x - 3$, $16x^3 + 8x^2 + x + 3$.
9. $2x^3 + 9x^2 - 17x - 45$, $6x^3 - 29x^2 + 31x + 10$.
10. $x^4 - 6x^3 + 8x^2 - 11x + 2$, $2x^4 - 11x^3 + 8x^2 - 6x + 1$.
11. $6x^3 + 11x^2 - 31x + 14$, $4x^3 - 47x + 7$.
12. $5x^3 + 12x^2 + 3x - 2$, $x^5 + 3x^4 + x^3 - x^2 - 4$.
13. $4x^3 - 17x^2 + 3x + 4$, $x^3 - 17x + 4$.
14. $2x^3 - 7x^2 - 46x - 21$, $2x^4 + 11x^3 - 13x^2 - 99x - 45$.
15. $15x^3 + 6x^2 - 45x - 18$, $-49x^3 + 28x^2 + 147x - 84$.
16. $6x^4 - 25x^2y^2 - 9y^4$, $3x^3 - 15x^2y + xy^2 - 5y^3$.
17. $3x^4 + 3x^3y - 27x^2y^2 + 33xy^3 - 12y^4$, $5x^4 - 5x^3y - 15x^2y^2 + 25xy^3 - 10y^4$.
18. $25x^4 + 5x^3 - x - 1$, $20x^4 + x^2 - 1$.
19. $x^3 + 4x^2 + 5x + 6$, $x^4 + 2x^3 + 5x^2 + 4x + 4$.
20. $3x^3 + 17x^2 - 62x + 14$, $7x^3 + 52x^2 - 46x + 8$.

REDUCTION OF FRACTIONS TO LOWEST TERMS.

116. We shall assume throughout that as the symbols stand for numerical quantities, the ordinary Arithmetical rules concerning Vulgar Fractions apply to Algebra, leaving the proofs of those rules to a later stage.

In Arithmetic $\dfrac{6}{8} = \dfrac{3 \times 2}{4 \times 2} = \dfrac{3}{4}$.

So in Algebra $\dfrac{ma}{mb} = \dfrac{a}{b}$.

$$\frac{abc^2}{b^2c} = \frac{ac \times bc}{b \times bc} = \frac{ac}{b}.$$

$$\frac{ax - bx}{abx} = \frac{(a - b) \times x}{ab \times x} = \frac{a - b}{ab}.$$

$$\frac{4a^2 - 6ab}{6a^2 - 4ab} = \frac{2a(2a - 3b)}{2a(3a - 2b)} = \frac{2a - 3b}{3a - 2b}.$$

$$\frac{x^2 - 5x + 6}{x^2 - 4x + 4} = \frac{(x - 2)(x - 3)}{(x - 2)^2} = \frac{(x - 2)(x - 3)}{(x - 2)(x - 2)} = \frac{x - 3}{x - 2}.$$

117. A fraction is reduced to its lowest terms by dividing its numerator and denominator by their H.C.F.

The H.C.F. should always be found by factorization, when possible.

Reduce $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}$ *to its lowest terms.*

$$\text{The given expression} = \frac{(3x-1)(x+1)}{x^2(x+1)-(x+1)}$$
$$= \frac{(3x-1)(x+1)}{(x^2-1)(x+1)}$$
$$= \frac{3x-1}{x^2-1} \text{ in its lowest terms.}$$

Reduce $\dfrac{a^3-7a^2+16a-12}{3a^3-14a^2+16a}$ *to its lowest terms.*

The denominator $= a(3a^2-14a+16) = a(3a-8)(a-2)$.

Hence it is evident that if the numerator and denominator have a common factor, it is $a-2$.

Acting on this knowledge, we write the numerator to show $a-2$ as a factor, thus:

$$(a^3-2a^2)-(5a^2-10a)+6a-12$$
$$= a^2(a-2)-5a(a-2)+6(a-2)$$
$$= (a^2-5a+6)(a-2)$$
$$= (a-2)(a-3)(a-2);$$

$\therefore$ the given expression $= \dfrac{(a-2)(a-3)(a-2)}{a(3a-8)(a-2)} = \dfrac{(a-2)(a-3)}{a(3a-8)}$ in its lowest terms.

Examples. XIX. d.

Reduce the following to their lowest terms:

1. $\dfrac{4a^3}{8a}$.
2. $\dfrac{10x^3}{5ax}$.
3. $\dfrac{10a^2b^2c}{24ab^2c^2}$.
4. $\dfrac{18x^5y^2z^3}{24x^3y^4z}$.
5. $\dfrac{18ab^4c^3}{12a^3b^2c^3}$.
6. $\dfrac{105m^3n^4p^6}{42m^2n^6p^3}$.
7. $\dfrac{a^2}{a^2+ab}$.
8. $\dfrac{x^2}{x^2-xy}$.
9. $\dfrac{3ax}{4ax-3ay}$.
10. $\dfrac{3ax}{3ax^2-3axy}$.
11. $\dfrac{6a^2-9ab}{8ab-12b^2}$.
12. $\dfrac{8x^2-12xy}{6x^2-4xy}$.
13. $\dfrac{3x^4-3x^2y^2}{5x^4-5x^2y^2}$.
14. $\dfrac{abx-bx^2}{acx-cx^2}$.
15. $\dfrac{xy-xyz}{3bz-3bz^2}$.

Reduce the following to their lowest terms:

16. $\dfrac{x^2-2x}{x^2-5x+6}$.
17. $\dfrac{3x-x^2}{x^2-5x+6}$.
18. $\dfrac{x^2+4x+4}{x^2+5x+6}$.
19. $\dfrac{1+3x+2x^2}{1-2x-3x^2}$.
20. $\dfrac{x^2+(a+b)x+ab}{x^2+(a+c)x+ac}$.
21. $\dfrac{a^2-b^2}{a^3-b^3}$.
22. $\dfrac{x^2-2xy+y^2}{x^2-y^2}$.
23. $\dfrac{b^2-a^2}{a^2+2ab+b^2}$.
24. $\dfrac{1+(a+b)x+abx^2}{1+(a+c)x+acx^2}$.
25. $\dfrac{2x^2-18}{3x^2+3x-18}$.
26. $\dfrac{x^4-3x^2+2}{x^4-x^2-2}$.
27. $\dfrac{x^2-(a-b)x-ab}{x^2-(a+c)x+ac}$.
28. $\dfrac{x^6-2x^3y^3+y^6}{x^6-y^6}$.
29. $\dfrac{x^2-7x+10}{2x^2-x-6}$.
30. $\dfrac{a^2+2ab+b^2-c^2}{a^2-b^2-2bc-c^2}$.
31. $\dfrac{3x^2+2x-1}{x^3+x^2-x-1}$.
32. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$.
33. $\dfrac{x^2-x-20}{x^2+x-12}$.
34. $\dfrac{x^4+x^2+1}{x^2+x+1}$.
35. $\dfrac{x^3+4x^2-5}{x^3-3x+2}$.*
36. $\dfrac{x^2-1}{3x^3+7x-10}$.
37. $\dfrac{x^4-9a^2}{x^4-6ax^2+9a^2}$.
38. $\dfrac{x^3+4x^2-5x}{x^3-6x+5}$.*
39. $\dfrac{(x-y)^2-1}{(x+1)^2-y^2}$.
40. $\dfrac{a^3+a^2+a-3}{a^3+3a^2+5a+3}$.*†
41. $\dfrac{3a^2-7ab+4b^2}{3a^2-ab-2b^2}$.
42. $\dfrac{4-(x+y)^2}{(x+2)^2-y^2}$.
43. $\dfrac{6a^2-13ab+6b^2}{6a^2-5ab-6b^2}$.
44. $\dfrac{(2a+b)^2-c^2}{4a^2-(b+c)^2}$.
45. $\dfrac{27+a^3}{9+3a}$.
46. $\dfrac{3x^2+5x+2}{3x^2+x-2}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

118. Example 1. Simplify $\dfrac{ab-ac}{ab-bc}\times\dfrac{3abc}{12a^2}\times\dfrac{a^2-ac}{b^2-bc}$.

The given expression $=\dfrac{a(b-c)}{b(a-c)}\times\dfrac{bc}{4a}\times\dfrac{a(a-c)}{b(b-c)}$

(factorizing and dividing numerator and denominator by $3a$)

$$=\frac{a^2bc(b-c)(a-c)}{4ab^2(a-c)(b-c)}$$

$$=\frac{ac}{4b}$$

[for a, b, $(b-c)$, $(a-c)$ are all common factors of numerator and denominator].

Example 2. Simplify $\dfrac{x^2+x-2}{x^2-2x}\times\dfrac{x^2-x-2}{x^2-2x-8}\div\dfrac{x^2-1}{x^2-5x}$.

The given expression $=\dfrac{(x+2)(x-1)}{x(x-2)}\times\dfrac{(x-2)(x+1)}{(x-4)(x+2)}\times\dfrac{x(x-5)}{(x-1)(x+1)}$

$$=\frac{x-5}{x-4}.$$

*The sum of the numerical coefficients is zero. ∴ $x-1$ is a factor. (Art. 95.)

†The sum of the coefficients of even powers = the sum of the coefficients of odd powers. (Art. 95.)

Examples. XIX. e.

Simplify the following:

1. $\dfrac{x^2-y^2}{x^2+2xy+y^2} \times \dfrac{xy+y^2}{x^2-xy}$.

2. $\dfrac{x^2-49}{x^2-9} \div \dfrac{x+7}{x+3}$.

3. $\dfrac{x^2-4}{2x-4} \times \dfrac{2}{x+2}$.

4. $\dfrac{4x^2-1}{4y^2-1} \div \dfrac{2x+1}{2y-1}$.

5. $\dfrac{x^2-5x+6}{x^2-16} \times \dfrac{x^2+5x+4}{x^2-4} \div \dfrac{x-3}{x-4}$.

6. $\dfrac{x^2+(a+b)x+ab}{x^2-c^2} \div \dfrac{x+a}{x-c}$.

7. $\dfrac{x^2+5x+6}{x^2-25} \div \dfrac{x+3}{x-5}$.

8. $\dfrac{x^4-a^4}{x^2-2ax+a^2} \div \dfrac{x^2+a^2}{a(x-a)}$.

9. $\dfrac{25a^2-1}{9x^2-4y^2} \times \dfrac{3x+2y}{5a+1} \div \dfrac{5a-1}{3x-2y}$.

10. $\dfrac{x^2-x-6}{x^2+x-2} \div \dfrac{x^2-3x}{x^2-x}$.

11. $\dfrac{6x^2+5x+1}{6x^2-x-1} \times \dfrac{2x^2-11x+5}{2x^2-11x-6}$.

12. $\dfrac{x^4-27x}{x^2-9} \div \dfrac{x^2+3x+9}{x+3}$.

13. $\dfrac{2x^2+x-1}{2x^2-x-1} \div \left(\dfrac{x^2+4x+3}{x^2+4x-5} \times \dfrac{2x-1}{2x+1}\right)$.

14. $\dfrac{x^2-5x+6}{x^2-10x+21} \times \dfrac{3(x^2-49)}{x^2+5x-14} \div \dfrac{x^2+5x-6}{x^2-x}$.

15. $\dfrac{(a+b)^2-c^2}{a^2-(b+c)^2} \times \dfrac{(a-b)^2-c^2}{(a+b)^2-c^2}$.

16. $\dfrac{x^3-64}{x^2-16} \times \dfrac{(x-3)^2}{(x+4)^2-4x} \div \dfrac{x^2+2x-15}{4x^2+16x}$.

17. $\dfrac{8x^2+14x+3}{8x^2-10x+3} \times \dfrac{12x^2-6x}{4x^2+5x+1} \div \dfrac{18x^2-6x}{4x^2+x-3}$.

18. $\dfrac{(a^2+ax)^2}{(a^2-ax)^2} \times \dfrac{a^3-x^3}{a^3+x^3} \div \dfrac{a+x}{a-x}$.

19. $\dfrac{6x^2+6}{(x+1)^2-x} \times \dfrac{x^3-1}{x^3-3x^2} \times \dfrac{x^3+x^2}{x^4-1}$.

20. $\dfrac{(a-b)^2-c^2}{ab-b^2-bc} \times \dfrac{c}{a^2+ab-ac} \div \dfrac{ac-bc+c^2}{a^2-(b-c)^2}$.

21. $\dfrac{3x-6x^2}{1-9x+18x^2} \times \dfrac{1-8x^3}{(1-2x)^2} \div \dfrac{3+6x+12x^2}{1+3x-18x^2}$.

22. $\dfrac{x^3+216}{x^2-x-42} \times \dfrac{x^3-3x^2}{x^4-6x^3+36x^2} \div \dfrac{x^2+2x-15}{2x^2-98}$.

23. $\dfrac{x^4+x^2+1}{x^2-1} \times \dfrac{(x-1)^3}{x^3-1} \div \dfrac{x^3+8x^2-9x}{x+1}$.

24. $\dfrac{8x^2-26x+15}{3x^2-x-4} \times \dfrac{3x^2-7x+4}{2x^2-7x+5} \div \dfrac{4x^2+x-3}{x^2-1}$.

25. $\dfrac{x^4-a^4}{x^6+a^6} \times \dfrac{x^3+a^3}{x^4-2a^2x^2+a^4} \div \dfrac{(x^3-a^3)}{(x^4-a^2x^2+a^4)(x^2-2ax+a^2)}$.

26. $\dfrac{15x^2-31xy+14y^2}{10x^2+xy-21y^2} \times \dfrac{21x^2-9xy}{3x^2-2xy+3x-2y} \div \dfrac{27x^2-63xy}{2x^2+3xy+2x+3y}$.

CHAPTER XX.

LOWEST COMMON MULTIPLE.

119. The **lowest common multiple** (L.C.M.) of two or more integral algebraic expressions is the integral expression of the lowest degree which is exactly divisible by each of them.

The L.C.M. of a^3b^2 and ab^3 is a^3b^3.

a^2, a^7, a^2, a is a^7.

.... $12a^3$ and $18a^2$ is $36a^3$, for 36 is the L.C.M. of 12 and 18, and a^3 is the L.C.M. of a^3 and a^2.

Example 1. Find the L.C.M. of $21a^6b^3c$, $7a^3b^2c^4$, and $2a^2b^5c^3$.

The L.C.M. of 21, 7, and 2 is 42.

The L.C.M. of a^6b^3c, $a^3b^2c^4$, $a^2b^5c^3$

must contain a^6 or it would not be divisible by the first expression,

it .... b^5 .. .. third ....

and .... c^4 ....second ....

$\therefore$ $42a^6b^5c^4$ is the reqd. L.C.M.

Examples. XX. a.

Find the lowest common multiple of:

1. a^2bc, ab^2c.
2. ax^2, $4a^2x$.
3. $4a^3$, $6a^5$.
4. $6xy^2$, $15x^2y$.
5. $42x^3y$, $49y^2z$.
6. a^2, $2ab$, b^2.
7. $10x^4$, $12x^2y^2$, $4xy^3$.
8. xy, yz, zx.
9. $8a^3b$, $12a^2b^2$, $3ab^3$, $4b^4$.
10. a^4, $4a^3b$, $6a^2b^2$, $4ab^3$, b^4.
11. $9x^4y$, $12x^3y^2$, $54x^2y^3$.
12. ay^2, az^2, a^2y, a^2z.
13. a, $2a$, $3a$, $4a$, $5a$.
14. a^3b^3, a^2b^2, ab.
15. $6a^3b^2c^4$, $4ab^3c^2$, $9a^2b^4c$.
16. $8x^3y^4z^5$, $5x^5y^2z^5$, $12x^2y^4z^6$, $16x^6y^4z^2$.

120. The L.C.M. of *compound expressions* can be determined by inspection when the expressions have been resolved into their simplest factors.

Example 1. Find the L.C.M of $a^3b - a^2bx$ and $ab^2c - b^2cx$.

$$a^3b - a^2bx = a^2b\,(a - x),$$
$$ab^2c - b^2cx = b^2c\,(a - x).$$

Thus we see that the reqd. L.C.M is $a^2b^2c\,(a - x)$.

Example 2. Find the L.C.M. of $x^2 - 5x + 6$ and $x^2 + 2x - 8$.

$$x^2 - 5x + 6 = (x - 2)(x - 3),$$
$$x^2 + 2x - 8 = (x - 2)(x + 4);$$
$$\therefore\ (x - 2)(x - 3)(x + 4) \text{ is the reqd. L.C.M.}$$

Example 3. $4a^4b^2c + 4a^3b^2cx$, $6a^3bc^2 - 6a^2bc^2x$, and $3a^2b^3c - 3b^3cx^2$.

$$4a^4b^2c + 4a^3b^2cx = 4a^3b^2c\,(a + x),$$
$$6a^3bc^2 - 6a^2bc^2x = 6a^2bc^2\,(a - x),$$
$$3a^2b^3c - 3b^3cx^2 = 3b^3c\,(a^2 - x^2) = 3b^3c\,(a - x)(a + x);$$
$$\therefore\ 12a^3b^3c^2\,(a - x)(a + x) \text{ is the reqd. L.C.M.}$$

Examples. XX. b.

Find the least common multiple of

1. $4x$, $4(a - x)$. **2.** a^2, $a(a - b)$ **3.** $2(a - x)$, $3(a + x)$.

4. $3(a + b)$, $7(a + b)$. **5.** $a^2b(a - b)$, $ab^2(a - b)$. **6.** $xyz(x - y)$, xy.

7. $2x^2(x + y)$, $4xy$. **8.** $6(x - 1)$, $2(x + 1)$, $(x^2 - 1)$. **9.** a^2, $a^2 - ax$.

10. $2a^3 + 2a^2x$, $4ax$ **11.** $3a - 3b$, $5a - 5b$. **12.** $4(x - y)$, $3(x^2 - y^2)$.

13. x^2, $(x^2 + 1)^2$, $6(x^2 + 1)$ **14.** $3(ax - by)$, $4(ax + by)$, $6(a^2x^2 - b^2y^2)$.

15. $x(x^2 - y^2)$, $y(x + y)$, $x(x - y)$. **16.** $8(1 - x)$, $8(1 + x)$, $(1 + x^2)$.

17. $3(x^3 - 1)$, $4(x^2 + x + 1)$, $6(x - 1)$. **18.** $x^2 + 3x + 2$, $x^2 + 5x + 6$.

19. $x^2 - 2x + 1$, $x^2 + x - 2$. **20.** $x^2 - 9x + 14$, $x^2 - 10x + 21$.

21. $x^2 - 3x - 4$, $x^2 + 2x - 24$. **22.** $(a + b)^2 - c^2$, $(a + c)^2 - b^2$.

23. $6(x + y)^2$, $9(x + y)^3$. **24.** $2x^2 - 7x + 3$, $2x^2 + 5x - 3$

25. $3x^2 - 7x + 2$, $3x^2 + 8x - 3$ **26.** $x^2 - y^2$, $(x + y)^2$, $(x - y)^2$.

27. $x^2 - 36y^2$, $x^2 + 7xy + 6y^2$, $x^2 + 5xy - 6y^2$.

28. $7(a^2b + ab^2)$, $21(a^2 + ab)$, $35(b^2 - ab)$.

29. $3(x^2 - y^2)$, $6(x^2 + xy)$, $4(x^3 - x^2y)$.

30. $12x^2y(x^2 - 3x + 2)$, $18xy^2(x - 1)$, $8y^3(x - 2)^2$.

31. $a^3 - b^3$, $2a^2 - 3ab + b^2$, $a^3 + a^2b + ab^2$. **32.** $2x^2 - 7x + 3$, $3x^2 - 7x - 6$.

33. $x^2 - 5x + 6$, $x^2 - 2x - 3$, $x^2 - x - 2$. **34.** $x^2 - 4$, $x^2 - x - 2$, $x^3 + 2x^2 - x - 2$.

35. $6(a^4 - a^2b^2)$, $18ab(a^3 - b^3)$, $9b(a^3b + b^4)$.

36. $6x(x^3 - y^3)$, $9(x^3 - xy^2)$, $12(x^3 + 2xy^2 - 2x^2y - y^3)$.

37. $x^2 - 4a^2$, $x^3 + 2ax^2 + 4a^2x + 8a^3$, $x^3 - 2ax^2 + 4a^2x - 8a^3$.

38. $x^2 - (a + b)x + ab$, $x^2 + 3ax - 3ab - b^2$, $x^2 + (2a + b)x - ab - 3a^2$.

39. $4x^3 - 12x^2 - x + 3$, $2x^3 + x^2 - 18x - 9$.

40. $ab - b^2 - ca + bc$, $bc - c^2 - ab + ca$.

CHAPTER XXI.

ADDITION AND SUBTRACTION OF FRACTIONS.

121. We have already seen that, just as in Arithmetic

$$\frac{3}{7}+\frac{5}{7}=\frac{3+5}{7},$$

so in Algebra $$\frac{x}{a}+\frac{y}{a}=\frac{x+y}{a},$$

and $$\frac{x}{a}-\frac{y}{a}=\frac{x-y}{a}.$$

When in Arithmetic we wish to add or subtract fractions which have different denominators, the plan is to reduce all the fractions to *equivalent fractions having the same denominator.*

We adopt the same plan in Algebra.

Example 1. $\dfrac{x-3}{4}-\dfrac{x-2}{6}=\dfrac{3(x-3)}{3\times 4}-\dfrac{2(x-2)}{2\times 6}$

[12 is the L.C.M. of the denominators 4 and 6. We therefore multiply numerator and denominator in the first fraction by 3, and in the second by 2.]

$$=\frac{3(x-3)-2(x-2)}{12}=\frac{3x-9-2x+4}{12} \text{ (removing brackets)}$$

$$=\frac{x-5}{12} \text{ (collecting like terms).}$$

Example 2 Simplify $\dfrac{x+3}{3x}-\dfrac{4x-3}{4x^2}+\dfrac{5}{2x^3}$.

The given expression $=\dfrac{4x^2(x+3)}{4x^2\times 3x}-\dfrac{3x(4x-3)}{3x\times 4x^2}+\dfrac{6\times 5}{6\times 2x^3}$

(the L.C.M. of $3x$, $4x^2$, $2x^3$ is $12x^3$)

$$=\frac{4x^3+12x^2-12x^2+9x+30}{12x^3}$$

$$=\frac{4x^3+9x+30}{12x^3}.$$

Examples. XXI. a.

Simplify the following expressions:

1. $\dfrac{1}{x}+\dfrac{1}{3x}+\dfrac{1}{2x}$.
2. $\dfrac{a}{x}+\dfrac{a}{3x}-\dfrac{a}{2x}$.
3. $\dfrac{1}{a}+\dfrac{1}{b}+\dfrac{1}{c}$.
4. $\dfrac{1}{ax}+\dfrac{1}{bx}-\dfrac{1}{cx}$.
5. $\dfrac{a}{bc}+\dfrac{b}{ac}+\dfrac{c}{ab}$.
6. $\dfrac{x-3}{3}-\dfrac{x-4}{4}$.
7. $\dfrac{x}{6}-\dfrac{x+1}{7}$.
8. $\dfrac{2x-1}{3}-\dfrac{4x-8}{6}$.
9. $\dfrac{x-a}{a}-\dfrac{x-b}{b}$.

10. $\dfrac{3x-y}{xy}-\dfrac{3z-2y}{yz}$.

11. $\dfrac{x-3}{3x}-\dfrac{x-5}{5x}$.

12. $\dfrac{q+3r}{3qr}-\dfrac{2p-q}{2pq}$.

13. $\dfrac{x+1}{2}+\dfrac{x+2}{3}+\dfrac{x-4}{4}$.

14. $\dfrac{x+y}{5}-\dfrac{2x-7y}{10}+\dfrac{x-3y}{3}$.

15. $\dfrac{a-b}{b}-\dfrac{a+b}{a}+\dfrac{a^2+4ab+2b^2}{2ab}$.

16. $\dfrac{x-y}{xy}+\dfrac{y-z}{yz}+\dfrac{z-x}{zx}$.

17. $\dfrac{2c-a}{3c}-\dfrac{a-b}{2a}+\dfrac{3b}{4a}$.

18. $\dfrac{x-y}{x}+\dfrac{x^3+y^3}{xy^2}-\dfrac{x^2+y^2}{x^2}$.

19. $\dfrac{3b+4a}{2ab}+\dfrac{b-6c}{3bc}-\dfrac{a+6c}{4ac}$.

20. $\dfrac{2x+1}{3x}-\dfrac{3x+2}{5x}+\dfrac{1}{7}$.

21. $\dfrac{a^2-b^2}{a^2b^2}-\dfrac{c^2-b^2}{b^2c^2}+\dfrac{c^2-a^2}{a^2c^2}$.

22. $\dfrac{3x-6y}{3}-\dfrac{21x-14y}{7}+\dfrac{38x-57y}{19}$.

122. Note carefully the truth of the following statements

$$\frac{1}{2-x}=\frac{-1}{x-2}=-\frac{1}{x-2}.$$

This is obtained by multiplying numerator and denominator by -1

In the same way $\qquad \dfrac{a-b}{c-d}=\dfrac{b-a}{d-c}$,

and again $\qquad \dfrac{4x-3y}{y-x}=-\dfrac{4x-3y}{x-y}$

Example 1
$$\frac{7a}{a-b}-\frac{3a-2b}{b-a}=\frac{7a}{a-b}+\frac{3a-2b}{a-b}$$
$$=\frac{7a+3a-2b}{a-b}=\frac{10a-2b}{a-b}$$

Example 2 Simplify $\dfrac{x+3y}{x+y}-\dfrac{x-6y}{x+2y}$

The L C M of $x+y$ and $x+2y$ is $(x+y)(x+2y)$

Multiplying numerator and denominator in the first fraction by $x+2y$,

and in the second by $x+y$,

$$\text{the given expression}=\frac{(x+2y)(x+3y)}{(x+2y)(x+y)}-\frac{(x+y)(x-6y)}{(x+2y)(x+y)} \qquad (a)$$
$$=\frac{(x+2y)(x+3y)-(x+y)(x-6y)}{(x+2y)(x+y)}$$
$$=\frac{x^2+5xy+6y^2-(x^2-5xy-6y^2)}{(x+2y)(x+y)}$$
$$=\frac{x^2+5xy+6y^2-x^2+5xy+6y^2}{(x+2y)(x+y)} \qquad (b)$$
$$=\frac{10xy+12y^2}{(x+2y)(x+y)}=\frac{2y(5x+6y)}{(x+2y)(x+y)}.$$

The above example is worked out in full. After a little practice such steps as (*a*) and (*b*) may be omitted.

The common denominator should generally be left in factors, and the result reduced to its lowest terms.

Example 3. Simplify $\dfrac{a^2-b^2}{ab+b^2}-\dfrac{a-b}{a+b}$.

$$\text{The given expression}=\frac{(a-b)(a+b)}{b(a+b)}-\frac{a-b}{a+b}$$

$$=\frac{a-b}{b}-\frac{a-b}{a+b}$$

$$=(a-b)\left[\frac{1}{b}-\frac{1}{a+b}\right]$$

$$=(a-b)\frac{a+b-b}{b(a+b)}$$

$$=\frac{a(a-b)}{b(a+b)}.$$

Examples. XXI. b.

Express the following in their simplest forms:

1. $\dfrac{1}{x+1}+\dfrac{1}{x-1}$.

2. $\dfrac{3}{x-1}+\dfrac{1}{1-x}$.

3. $\dfrac{\quad}{x+3}+\dfrac{\quad}{x+4}$.

4. $\dfrac{1}{x+3}-\dfrac{1}{x+4}$.

5. $\dfrac{6}{2x-3y}-\dfrac{3}{3y-2x}$.

6. $\dfrac{4}{x+6}-\dfrac{2}{x+3}$.

7. $\dfrac{3}{3x-1}-\dfrac{2}{2x+3}$.

8. $\dfrac{x}{x+y}+\dfrac{y}{x-y}$.

9. $\dfrac{x+2}{x+4}-\dfrac{x+5}{x+10}$.

10. $\dfrac{x+5}{x-2}-\dfrac{x-5}{2-x}$.

11. $\dfrac{x+3}{x-3}-\dfrac{x-3}{x+3}$.

12. $\dfrac{3}{1-x}+\dfrac{4}{(1-x)^2}$.

13. $\dfrac{2x-1}{x+1}-\dfrac{2x-1}{x-1}$.

14. $\dfrac{1}{x-y}+\dfrac{2x-y}{x^2-y^2}$.

15. $\dfrac{4x}{(x+y)^2}-\dfrac{4}{x+y}$.

16. $\dfrac{1}{1-2x}-\dfrac{2x}{1-4x^2}$.

17. $\dfrac{3a}{9a^2-4b^2}-\dfrac{1}{3a+2b}$.

18. $\dfrac{2y}{(x-2y)^2}+\dfrac{1}{x-2y}$.

19. $\dfrac{x}{x^2-y^2}+\dfrac{y}{y^2-x^2}$.

20. $\dfrac{\quad}{x-4}-\dfrac{16-3x}{x^2-16}$.

21. $\dfrac{x-y}{x^2-y^2}+\dfrac{1}{2x+3y}$.

(In the first fraction, $x-y$ is a common factor of numerator and denominator.)

22. $\dfrac{1}{y-x}+\dfrac{x}{(x-y)^2}$.

23. $\dfrac{a-b}{c-d}-\dfrac{b-a}{d-c}$.

24. $\dfrac{2a-b}{c-d}-\dfrac{a-2b}{d-c}$.

25. $\dfrac{1}{a(a-b)}+\dfrac{1}{b(a+b)}$.

26. $\dfrac{2x}{a^2-4x^2}-\dfrac{1}{2x+a}$.

27. $\dfrac{5}{3(a-b)}+\dfrac{3}{2(b-a)}$.

28. $\dfrac{x+a}{x-a}+\dfrac{x^2-a^2}{ax-a^2}$.

29. $\dfrac{a}{a^2-9b^2}+\dfrac{1}{3b-a}$.

30. $\dfrac{a+3b}{a-2b}-\dfrac{2a+6b}{2a+5b}$.

31. $\dfrac{1}{a^3-1}+\dfrac{a+1}{a^2+a+1}$.

32. $\dfrac{1}{x-2y}-\dfrac{x^2+4y^2}{x^3-8y^3}$.

33. $\dfrac{1}{9a^2-3ab+b^2}-\dfrac{3a}{27a^3+b^3}$.

34. $\dfrac{a^2-4b^2}{a-2b}-\dfrac{a^2-9b^2}{a+3b}$.

35. $\dfrac{x^3+y^3}{x^2-xy+y^2}+\dfrac{x^3-y^3}{x^2+xy+y^2}$.

36. $\dfrac{x^2+5x+4}{x+4}-\dfrac{x^2-5x+6}{x-2}$.

37. $\left(\dfrac{x+y}{x-y}\right)^2-1$.

38. $\dfrac{x-2}{x^2-x-2}+\dfrac{x-4}{x^2-5x+4}$.

39. $\dfrac{x+4}{x^2-3x-28}-\dfrac{x-5}{x^2+2x-35}$.

40. $\dfrac{x^2-4y^2}{x^2+2xy}-\dfrac{x-2y}{x}$.

41. $\dfrac{6x+5y}{4}-\dfrac{9x^2-y^2}{6x+2y}$.

***Example 1** $\dfrac{a}{a-b}-\dfrac{b}{a+b}-\dfrac{b^2}{a^2-b^2}-\dfrac{a^2}{a^2+b^2}$

$$=\frac{a(a+b)-b(a-b)-b^2}{a^2-b^2}-\frac{a^2}{a^2+b^2}$$

(taking the first three fractions together)

$$=\frac{a^2+ab-ab+b^2-b^2}{a^2-b^2}-\frac{a^2}{a^2+b^2}$$

$$=\frac{a^2}{a^2-b^2}-\frac{a^2}{a^2+b^2}$$

$$=a^2\left(\frac{1}{a^2-b^2}-\frac{1}{a^2+b^2}\right)$$

$$=\frac{a^2(a^2+b^2-a^2+b^2)}{a^4-b^4}$$

$$=\frac{2a^2b^2}{a^4-b^4}$$

***Example 2** Simplify $\dfrac{3}{x-a}-\dfrac{1}{x-3a}-\dfrac{3}{x+a}+\dfrac{1}{x+3a}$

The given expression $=\left(\dfrac{3}{x-a}-\dfrac{3}{x+a}\right)+\left(\dfrac{1}{x+3a}-\dfrac{1}{x-3a}\right)$ (rearranging the fractions)

$$=\frac{3x+3a-3x+3a}{x^2-a^2}+\frac{x-3a-x-3a}{x^2-9a^2}$$

$$=\frac{6a}{x^2-a^2}-\frac{6a}{x^2-9a^2}$$

$$=6a\left(\frac{1}{x^2-a^2}-\frac{1}{x^2-9a^2}\right)$$

$$=\frac{6a(x^2-9a^2-x^2+a^2)}{(x^2-a^2)(x^2-9a^2)}$$

$$=\frac{-48a^3}{(x^2-a^2)(x^2-9a^2)}$$

*Examples. XXI. c.

Simplify

1. $\dfrac{1}{a+b}+\dfrac{1}{a-b}+\dfrac{2a}{a^2-b^2}$.

2. $\dfrac{1}{a+b}+\dfrac{1}{b-a}+\dfrac{4b}{a^2-b^2}$.

3. $\dfrac{1}{1-3x}+\dfrac{1}{1+3x}+\dfrac{1}{1-9x^2}$.

4. $\dfrac{1}{x^2-5x+4}-\dfrac{1}{x^2-4x+3}$.

Simplify

5. $\frac{a}{a^2-b^2}-\frac{1}{3(a-b)}-\frac{1}{3(a+b)}.$

6. $\frac{1}{3(x-3)}+\frac{1}{x^2-9}-\frac{1}{2(x+3)}.$

7. $\frac{1}{x-1}-\frac{2}{x-2}+\frac{1}{x-3}.$

8. $\frac{a^2}{a^3+b^3}+\frac{a-b}{a^2-ab+b^2}+\frac{1}{a+b}.$

9. $\frac{1}{x-3}-\frac{8x}{x^3-27}-\frac{x-3}{x^2+3x+9}.$

10. $\frac{ab}{(a-b)(b-c)}+\frac{ac}{(a-c)(c-b)}.$

11. $\frac{1}{x^2-4x+3}+\frac{1}{x^2-5x+6}+\frac{1}{x^2-3x+2}.$

12. $\frac{1}{x-2y}+\frac{2(x+1)}{2x-y}-\frac{1+2x}{2x-y}.$

13. $\frac{1}{6x-2}-\frac{1}{2x-\frac{2}{3}}+\frac{1}{3x-1}.$

14. $\frac{3x}{x^2-3x+2}+\frac{4}{1-x}+\frac{1}{x-2}.$

15. $\frac{1}{2(x-2)}+\frac{1}{(x-2)(x-3)}+\frac{1}{(x-2)(x-3)(x-4)}.$

16. $\frac{4y}{x^2+2xy}-\frac{3x}{xy+2y^2}+\frac{3x-2y}{xy}.$

17. $\frac{1}{(x-2)(x-3)}-\frac{1}{x^2+x-6}-\frac{3}{9-x^2}.$

18. $\frac{a^2-3ab+2b^2}{a-2b}+\frac{6a^2-5ab-6b^2}{2a-3b}-\frac{6a^2+ab-2b^2}{3a+2b}.$

19. $\frac{a-b}{a^2-ab+b^2}+\frac{1}{a+b}+\frac{ab}{a^3+b^3}.$

20. $\frac{2}{x^2-8x+15}+\frac{2}{x^2-4x+3}+\frac{4}{6x-x^2-5}.$

21. $\frac{1}{x^4+2x^3}+\frac{1}{x^4-2x^3}+\frac{2}{x^4+4x^2}.$

22. $\frac{1}{x-1}+\frac{2}{x+1}+\frac{3x-2}{1-x^2}-\frac{1}{(x+1)^2}.$

23. $\frac{x+1}{2x^3-4x^2}+\frac{x-1}{2x^3+4x^2}-\frac{1}{x^2-4}.$

24. $\frac{8}{x^2-5x+6}-\frac{5}{x^2-3x+2}-\frac{3}{x^2-4x+3}.$

25. $\frac{x^3y-xy^3}{x^6-y^6}+\frac{x}{x^3-y^3}-\frac{y}{x^3+y^3}.$

26. $\frac{1}{x^2-x-2}+\frac{2}{1-x^2}+\frac{1}{x^2+x-2}.$

27. $\frac{2y}{x^2+xy-6y^2}+\frac{x}{x^2-9y^2}-\frac{1}{x-2y}.$

28. $\frac{5}{x^2-3x-28}+\frac{3}{x^2+x-12}+\frac{9}{x^2-10x+21}.$

29. $\frac{4}{x+3}-\frac{7}{x+4}+\frac{3}{x+7}.$

30. $\frac{1}{4(3a-x^2)}-\frac{1}{5(3a+x^2)}-\frac{9x^2}{10(9a^2-x^4)}.$

31. $\frac{1}{2x^2-4x+2}-\frac{1}{3x^2-3}+\frac{1}{4x^2+8x+4}.$

32. $\frac{x-3y}{x+3y}-\frac{x-2y}{x+2y}+2.$

33. $\frac{1+x^2}{1-x^2}-\frac{4x^2}{1-x^4}-\frac{1-x^2}{1+x^2}.$

34. $\frac{5x}{3x-2}-\frac{21x^2+6x}{9x^2+4}+\frac{2x}{3x+2}.$

35. $\frac{b}{a-b}-\frac{8b}{a-2b}+\frac{9b}{a-3b}.$

36. $\frac{1}{x^2+2xy-3y^2}+\frac{1}{y^2+2xy-3x^2}.$

37. $\frac{1+x^2}{1-x^2}+\frac{4x^2}{1+x^4}-\frac{1-x^2}{1+x^2}.$

38. $\frac{1}{a^2-2}-\frac{2}{a^2-1}+\frac{2}{a^2+1}-\frac{1}{a^2+2}.$

39. $\dfrac{x^2-7xy+12y^2}{4x^2-11xy-3y^2}-\dfrac{2x^2+7xy-4y^2}{8x^2-6xy+y^2}$.

40. $\dfrac{x}{x-y}-\dfrac{y}{x+y}-\dfrac{x^2}{x^2+y^2}+\dfrac{y^2}{y^2-x^2}$.

41. $\dfrac{4a^2b^2}{a^4-b^4}+\dfrac{2a^2}{a^2+b^2}+\dfrac{a}{a+b}-\dfrac{a}{b-a}$.

42. $\dfrac{(2a-5b)^2-4a^2}{4a-5b}+\dfrac{(3a-2b)^2-4b^2}{3a-4b}$.

43. $\dfrac{6x^2-5xy-6y^2}{14x^2-23xy+3y^2}-\dfrac{15x^2+8xy-12y^2}{35x^2+47xy+6y^2}$.

44. $\dfrac{x}{x-y-z}+\dfrac{y}{y+z-x}-\dfrac{x+y}{x+y+z}$.

45. $\dfrac{1}{a-5}-\dfrac{1}{a-3}+\dfrac{1}{a+5}-\dfrac{1}{a+3}$.

46. $\dfrac{1}{x+1}+\dfrac{2}{x^2+1}+\dfrac{4}{x^4+1}+\dfrac{8}{x^8-1}$.

47. $\dfrac{1}{a-b}-\dfrac{1}{2(a+b)}-\dfrac{a+3b}{2(a^2+b^2)}-\dfrac{4b^3}{a^4-b^4}$.

48. $\dfrac{5}{3-2x}-\dfrac{15}{(3-2x)^2}+\dfrac{30x}{(3-2x)^3}$.

49. $\dfrac{1+a}{1-a}+\dfrac{4a}{1+a^2}+\dfrac{8a}{1-a^4}-\dfrac{1-a}{1+a}$.

50. $\dfrac{3x^2+2x+4}{x^3-1}-\dfrac{x+1}{x^2+x+1}-\dfrac{2}{x-1}$.

51. $\dfrac{4}{x(x-2)}+\dfrac{1}{x^2-5x+6}-\dfrac{3}{x(x-3)}$.

52. $\dfrac{1}{x-1}-\dfrac{3}{x+1}+\dfrac{2(x-2)}{x^2+1}$.

53. $\dfrac{1}{a^2-3b^2+2ab}+\dfrac{1}{b^2-3a^2+2ab}-\dfrac{2}{3a^2+10ab+3b^2}$.

54. $\dfrac{2x+1}{x^2+x+1}-\dfrac{3}{x}-\dfrac{1}{1-x}$.

55. $\dfrac{b}{a+b}-\dfrac{ab}{(a+b)^2}-\dfrac{ab^2}{(a+b)^3}$.

56. $\dfrac{8x^3}{8x^3-y^3}-\dfrac{2x^2}{4x^2+2xy+y^2}+\dfrac{x}{y-2x}$.

57. $\dfrac{1}{x+4}-\dfrac{3}{x+3}+\dfrac{3}{x+2}-\dfrac{1}{x+1}$.

58. $\dfrac{2}{x+1}+\dfrac{3}{(x+1)^2}+\dfrac{2x+5}{x^2-2x+3}$.

59. $\dfrac{1}{2(x-1)}-\dfrac{x-5}{x^2-7x+10}+\dfrac{x-6}{2(x^2-9x+14)}$.

60. $\dfrac{x}{x^2+y^2-xy}+\dfrac{1}{x+y}+\dfrac{2xy-y^2}{x^3+y^3}$.

61. $\dfrac{1}{x-3}+\dfrac{1}{x+3}-\dfrac{1}{x-1}-\dfrac{1}{x+1}$.

62. $\dfrac{10x-11}{3(x^2-1)}-\dfrac{10x-1}{3(x^2+x+1)}+\dfrac{x^2-2x+5}{(x^3-1)(x+1)}$.

63. $\dfrac{3(x^2+x-2)}{x^2-x-2}-\dfrac{3(x^2-x-2)}{x^2+x-2}-\dfrac{8x}{x^2-4}$.

64. $\dfrac{a+2}{a}-\dfrac{a}{a+2}-\dfrac{a^3-2a^2}{2a^2-8}$.

65. $\dfrac{2}{x^2+x}+\dfrac{2x-1}{x^2-x+1}-\dfrac{2x^3-1}{x^4+x}$.

66. $\dfrac{2x+9}{x^2+7x+12}-\dfrac{x}{x^2+5x+6}-\dfrac{x}{x^2+3x+2}$.

67. $\dfrac{a}{b}-\dfrac{(a^2-b^2)x}{b^2}+\dfrac{a(a^2-b^2)x^2}{b^2(b+ax)}$.

68. $\dfrac{3}{3x-2}-\dfrac{2}{2x+1}-\dfrac{3}{4-3x}$.

69. $\dfrac{a-2b}{2a^2-11ab+12b^2}+\dfrac{2(2a-b)}{4a^2-4ab-3b^2}-\dfrac{3(a-b)}{2a^2-7ab-4b^2}$.

70. $\dfrac{\dfrac{1}{1+x}}{1-\dfrac{1}{1+x}}$.

71. $\dfrac{x+y-\dfrac{x^2+y^2}{x+y}}{x+y-\dfrac{2xy}{x+y}}$.

72. $\dfrac{\dfrac{1}{a+b}+\dfrac{1}{a-b}}{\dfrac{1}{a+b}-\dfrac{1}{a-b}}$.

Simplify:

73. $\frac{x^2+3x+2}{(x-1)^2}\left\{1-\frac{3(3x+2)}{3x^2+8x+4}\right\}$. **74.** $\frac{1-\left(\frac{x-y}{x+y}\right)^2}{1+\left(\frac{x-y}{x+y}\right)^2}$. **75.** $\frac{x-2-\frac{x^2-5x}{x-3}}{x+\frac{3x}{x-3}}$.

76. $\left(\frac{x+3}{x^2-4}+\frac{x+5}{x^3+8}\right)\div\frac{x^2+1}{x^2-2x+4}$. **77.** $\frac{a+x-\frac{a^3}{a^2-ax+x^2}}{a+x-\frac{x^3}{a^2-ax+x^2}}$.

78. $\frac{a}{b}\left(\frac{b}{c}-\frac{c}{a}\right)+\frac{b}{c}\left(\frac{c}{a}-\frac{a}{b}\right)+\frac{c}{a}\left(\frac{a}{b}-\frac{b}{c}\right)$. **79.** $\frac{\frac{a^2+b^2}{b}+a}{\frac{1}{b}+\frac{1}{a}}\div\frac{b^3-a^3}{a^2-b^2}$.

80. $\frac{(a+3b)^2-(a-3b)^2}{(3a+b)^2-(3a-b)^2}$. **81.** $\frac{a^3-b^3}{a-b}-\frac{a^3+b^3}{a+b}+(a-b)^2$.

82. $\frac{\frac{m^2+n^2}{n}-m}{\frac{1}{m}-\frac{1}{n}}\div\frac{m^3+n^3}{m^2-n^2}$. **83.** $\frac{a-\frac{b^2}{a}}{\frac{a^3}{b^2}-\frac{b^2}{a}}\times\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$.

84. $\left\{\frac{x+2a}{a-2x}-\frac{a+2x}{x-2a}\right\}\times\left\{\frac{3}{2a-x}-\frac{1}{a-x}\right\}$.

85. $\left\{\frac{5a}{a-6b}-\frac{2b}{3a-2b}\right\}\div\left\{\frac{2a}{a+2b}-\frac{2b-a}{2b-3a}\right\}$.

86. $\frac{1}{x-\frac{3}{x-2}}-\frac{1}{x+\frac{2}{x+3}}$. **87.** $\frac{x(x+1)(x+2)}{3}-\frac{x(x+1)(2x+1)}{6}$.

88. $\left(\frac{x}{x-2}+\frac{5}{x-8}\right)\times\left(\frac{x-3}{3x-8}-\frac{2}{x+2}\right)$.

89. $\left(\frac{x}{x-y}-\frac{y}{x+y}\right)(x^2+2xy-y^2)\div\left(\frac{x}{x-y}+\frac{y}{x+y}\right)$.

90. $\left(1-\frac{2xy}{x^2+y^2}\right)\div\left(\frac{x^3-y^3}{x-y}-3xy\right)$. **91.** $\left(\frac{x+1}{x-1}+\frac{5}{x-7}\right)\left(\frac{x-2}{3x-5}-\frac{2}{x+3}\right)$.

92. $\left(\frac{a^3+b^3}{a^3-b^3}+\frac{a^3-b^3}{a^3+b^3}\right)\div\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)$.

93. $\frac{(2a+3)(a^2+3a+2)-2(a+1)(a^2+2a)}{(2a+3)a^2-a(a^2-2)}$.

94. $\frac{3}{2x+3-\frac{3}{1-\frac{x}{x+6}}}$. **95.** $\frac{1+\frac{4a^2}{6ab+9b^2}}{1+\frac{9b^2}{4a^2-6ab}}\div\left(\frac{16a^4}{81b^4}-\frac{2a}{3b}\right)$.

96. $1-\frac{1}{\frac{x}{a}+3+\frac{1}{x-2a}\left(4a+\frac{a^3-x^3}{x^2+ax+a^2}\right)}$.

97. $\dfrac{\frac{a-x}{b+x}-\frac{b-x}{a+x}}{\frac{a+x}{b+x}-\frac{b-x}{a-x}}-\dfrac{\frac{a+x}{b+x}-\frac{b-x}{a-x}}{\frac{a+x}{b-x}-\frac{b+x}{a-x}}$.

98. $\left(1+\frac{45}{x-8}-\frac{26}{x-6}\right)\left(3-\frac{65}{x+7}+\frac{8}{x-2}\right)$.

99. $\left\{\frac{x-a}{(x+a)^2}+\frac{x+a}{(x-a)^2}\right\}\div\left\{\frac{1}{(x+a)^2}-\frac{1}{x^2-a^2}+\frac{1}{(x-a)^2}\right\}$.

100. $\dfrac{1-\frac{1}{1+\frac{x}{1-x}}}{1-\frac{1}{1-\frac{x}{1+x}}}$.

101. $\left(\frac{1}{1-x^2}+\frac{1}{1-x}-1\right)\frac{(1-x)^2}{1-x^3}$.

102. $\dfrac{\frac{x+y}{x-y}-\frac{x-y}{x+y}}{\frac{x+y}{x-y}+\frac{x-y}{x+y}}-\dfrac{\frac{x}{y}-\frac{y}{x}}{\frac{x}{y}+\frac{y}{x}}$.

103. $(a+b+c)\left(\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}\right)-\frac{1}{abc}(a^2+b^2+c^2)$.

104. $\dfrac{(ac+bd)^2-(ad+bc)^2}{(a-b)(c-d)}$.

105. $\dfrac{\frac{c}{a+b}-\frac{a}{b+c}}{\frac{a}{b+c}-\frac{b}{c+a}}$.

106. $\left\{\left(x+\frac{1}{x}\right)^2-2\left(1+\frac{1}{x^2}\right)\right\}\div\left(x-\frac{1}{x}\right)^2$.

107. $\dfrac{\{ax^2+(b-c)x-f\}^2-\{ax^2+(b+c)x-f\}^2}{\{ax^2+(b+c)x-f\}^2-\{ax^2+(b-c)x-f\}^2}$.

108. $(yz+zx+xy)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-xy\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)$.

109. $\left(2-\frac{3n}{m}+\frac{9n^2-2m}{m^2+2mn}\right)\div\left(\frac{1}{m}-\dfrac{1}{m-2n-\frac{4n^2}{m+n}}\right)$.

110. $\left(x^2-1-\frac{6}{x^2}\right)\div\left(x^2-2x+3-\frac{4}{x}+\frac{2}{x^2}\right)$.

111. $\dfrac{a^2-(b-c)^2}{(c+a)^2-b^2}+\dfrac{b^2-(c-a)^2}{(a+b)^2-c^2}+\dfrac{c^2-(a-b)^2}{(b+c)^2-a^2}$.

112. $\left\{1+\frac{x^2-xy-y^2}{x^2+xy+y^2}\right\}\times\left\{\frac{1}{x}-\frac{y^3-xy^2}{x^4-x^2y^2}\right\}$.

113. $\dfrac{x^2-ax-2a^2}{x^2-(2a+b)x+2ab}-\dfrac{x^2+ax-2a^2}{x^2+(2a+b)x+2ab}$.

114. $\dfrac{9x^2-(y-z)^2}{(3x+z)^2-y^2}+\dfrac{y^2-(z-3x)^2}{(3x+y)^2-z^2}+\dfrac{z^2-(3x-y)^2}{(y+z)^2-9x^2}$.

115. $\left\{1+\frac{2b^2}{a(a-3b)}\right\}\left\{1+\frac{b}{2b-a}\right\}-\left(\frac{a^2}{b^2}+\frac{b}{a}\right)\left(\frac{a^2-ab}{a^2-ab+b^2}-1\right)$.

116. $\dfrac{\{(a+b)(a+b+c)+c^2\}\{(a+b)^2-c^2\}}{\{(a+b)^3-c^3\}\{a+b+c\}}$.

117. $\left(1+\frac{y^2+z^2-x^2}{2yz}\right)\div\left(1-\frac{x^2+y^2-z^2}{2xy}\right)$.

Simplify :

118. $\left(x-y-\dfrac{4y^2}{x-y}\right)\left(x+y-\dfrac{4x^2}{x+y}\right)\div\left\{3(x+y)-\dfrac{8xy}{x-y}\right\}$.

119. $\left(\dfrac{x^2}{y^2}-1\right)\left(\dfrac{x}{x-y}-1\right)\div\left(\dfrac{x^3}{y^3}-1\right)\left(\dfrac{x^2+xy}{x^2+xy+y^2}-1\right)$.

120. $\dfrac{1}{x+\dfrac{1}{x+2}}\times\dfrac{1}{x+\dfrac{1}{x-2}}\div\dfrac{x-\dfrac{4}{x}}{x^2+\dfrac{1}{x^2}-2}$. 121. $(1+a)^2\div\left\{1+\dfrac{a}{1-a+\dfrac{a}{1+a+a^2}}\right\}$.

Prove that

122. $\dfrac{a-2b}{a-b}+\dfrac{a-2b}{a+3b}-\dfrac{2(a+b)}{a+2b}=\dfrac{2b(a+b)(2a+b)}{(b-a)(a+3b)(a+2b)}$.

123. $\dfrac{a}{ax-x^2}+\dfrac{b}{bx-x^2}+\dfrac{c}{cx-x^2}=\dfrac{1}{a-x}+\dfrac{1}{b-x}+\dfrac{1}{c-x}+\dfrac{3}{x}$.

*CHAPTER XXII.

HARDER SIMPLE EQUATIONS INVOLVING FRACTIONS

123. The usual method of solution is to clear away the fractions by multiplying both sides of the equation by the L.C.M. of the denominators.

The work can often be shortened by sundry methods illustrated in the following worked out examples.

Example 1. Solve the equation $\dfrac{3}{4x-3}=\dfrac{2}{3x-5}$.

Multiplying both sides by $(4x-3)(3x-5)$, the L.C.M. of the denominators,

$$3(3x-5)=2(4x-3),$$
$$9x-15=8x-6,$$
$$x=9.$$

Example 2. Solve the equation $\dfrac{3x}{x-1}-\dfrac{2x}{x+1}=\dfrac{x^2+10}{x^2-1}$.

Multiplying both sides by $(x-1)(x+1)$,

$$3x(x+1)-2x(x-1)=x^2+10,$$
$$3x^2+3x-2x^2+2x=x^2+10,$$
$$5x=10,$$
$$x=2.$$

Example 3. Solve the equation $\dfrac{2}{2x-1}-\dfrac{3}{3x+1}=\dfrac{3}{3x-1}-\dfrac{2}{2x+1}$.

Simplifying each side of the equation separately,

$$\frac{2(3x+1)-3(2x-1)}{(2x-1)(3x+1)}=\frac{3(2x+1)-2(3x-1)}{(3x-1)(2x+1)},$$

$$\frac{6x+2-6x+3}{(2x-1)(3x+1)}=\frac{6x+3-6x+2}{(3x-1)(2x+1)},$$

$$\frac{5}{(2x-1)(3x+1)}=\frac{5}{(3x-1)(2x+1)}.$$

Dividing both sides by 5, and multiplying up,

$$(3x-1)(2x+1)=(2x-1)(3x+1),$$

$$6x^2+x-1=6x^2-x-1,$$

$$2x=0,$$

$$x=0.$$

Example 4. Solve the equation $\dfrac{10x-14}{2x-3}=\dfrac{15x-24}{3x-5}$.

The equation may be written $\dfrac{5(2x-3)+1}{2x-3}=\dfrac{5(3x-5)+1}{3x-5}$,

$$\text{i.e. } 5+\frac{1}{2x-3}=5+\frac{1}{3x-5};$$

$$3x-5=2x-3,$$

$$x=2.$$

Example 5. Solve the equation $\dfrac{x-3}{x-5}-\dfrac{x-1}{x-3}=\dfrac{x-7}{x-9}-\dfrac{x-5}{x-7}$.

The equation may be written

$$\frac{x-5+2}{x-5}-\frac{x-3+2}{x-3}=\frac{x-9+2}{x-9}-\frac{x-7+2}{x-7};$$

$$1+\frac{2}{x-5}-1-\frac{2}{x-3}=1+\frac{2}{x-9}-1-\frac{2}{x-7}.$$

Dividing both sides by 2,

$$\frac{1}{x-5}-\frac{1}{x-3}=\frac{1}{x-9}-\frac{1}{x-7}.$$

Simplifying each side separately,

$$\frac{(x-3)-(x-5)}{(x-5)(x-3)}=\frac{(x-7)-(x-9)}{(x-7)(x-9)},$$

$$\text{i.e. } \frac{2}{(x-5)(x-3)}=\frac{2}{(x-7)(x-9)}.$$

Dividing both sides by 2, and multiplying up,

$$(x-7)(x-9)=(x-5)(x-3),$$

$$x^2-16x+63=x^2-8x+15,$$

$$-8x=-48,$$

$$x=6.$$

*Examples. XXII.

(In the case of a fractional solution, express the result in decimals correct to two decimal places.)

Solve the equations:

1. $\frac{x-3}{x-4}=\frac{x+12}{x+8}$.

2. $\frac{x+3}{2x-3}=\frac{2x}{4x-9}$.

3. $3-\frac{22}{x+5}=\frac{6x-1}{2x+7}$.

4. $\frac{x}{x-3}+\frac{2}{x-5}=1$.

5. $\frac{x+1}{3x-4}=\frac{1}{5}+\frac{8x-3}{15x-20}$.

6. $\frac{6x-5}{8x-12}=\frac{1}{12}-\frac{3x-4}{6x-9}$.

7. $\frac{3}{x-3}+\frac{4}{x-4}=\frac{25}{x^2-7x+12}$.

8. $\frac{5x-7}{10x-5}=\frac{1}{10}-\frac{4x-3}{4x-2}$.

9. $\frac{11x}{x+20}+\frac{24}{x}=11+\frac{88}{x(x+20)}$.

10. $\frac{x-\frac{1}{2}}{x-1}-\frac{3}{5}\left(\frac{1}{x-1}-\frac{1}{3}\right)=\frac{23}{10(x-1)}$.

11. $\frac{9(12-x)}{4(x+1)}+\frac{5}{4}=\frac{17-x}{x-8}$.

12. $\frac{6x-7}{2x-3}-\frac{9x-12}{3x-5}=\frac{12x-25}{3x-7}-\frac{8x-18}{2x-5}$.

13. $\frac{x-4}{x-5}-\frac{x-2}{x-3}=\frac{x-10}{x-11}-\frac{x-8}{x-9}$.

14. $\frac{30+6x}{x+1}+\frac{60+8x}{x+3}=14+\frac{48}{x+1}$.

15. $\frac{6x+2}{x+15}+\frac{2x-9}{x-6}=6+\frac{2x-13}{x-6}$.

16. $\frac{3x-14}{x-5}-\frac{3x-8}{x-3}=\frac{3x-32}{x-11}-\frac{3x-26}{x-9}$.

17. $\frac{7x+1}{x-1}=\frac{35}{9}\left(\frac{x+4}{x+2}\right)+\frac{28}{9}$.

18. $\frac{\frac{x}{6}}{5x-4}=\frac{\frac{2x}{5}-\frac{27}{14}}{12x+\frac{33}{5}}$.

19. $\frac{x+2}{x-3}+\frac{x-2}{x-6}=2$.

20. $2\left(\frac{2x+3}{x-1}\right)+3\left(\frac{x-2}{x+2}\right)=7$.

21. $\frac{3x+2}{x-1}+\frac{2x-4}{x+2}=5$.

22. $\frac{8x}{2x-3}-\frac{5}{3x-2}=4$.

23. $\frac{1}{x+1}-\frac{2}{x+2}+\frac{1}{x+4}=0$.

24. $\frac{3x}{x-1}-\frac{2x}{2x-1}=2$.

25. $\frac{1}{x-1}+\frac{1}{x+1}=\frac{2}{x+2}$.

26. $\frac{1}{15}-\frac{1}{10x}-\frac{1}{15-6x}=\frac{1}{15x+120}$.

27. $\frac{4(2x-1)}{3(x-2)}-\frac{2(7x-1)}{6x-13}=\frac{1}{3}$.

28. $\frac{3x}{2x-3}-\frac{2x+17}{x+10}=\frac{1}{2}$.

29. $\frac{6x+1}{3x-5}-\frac{2x-5}{3x-4}=\frac{4}{3}$.

30. $\frac{1}{x-1}-\frac{1}{x-3}=3\left\{\frac{1}{x-2}-\frac{1}{x-3}\right\}$.

31. $\frac{10x+17}{18}-\frac{12x+2}{11x-8}=\frac{5x-4}{9}$.

32. $\frac{x-5}{x^2-6x+6}-\frac{x-7}{x^2-8x+15}=0$.

33. $\frac{x^2-x+1}{x-1}+\frac{x^2+x+1}{x+1}=2x$.

34. $\frac{x-1}{x-2}-\frac{x}{x-1}=\frac{x-8}{x-9}-\frac{x-7}{x-8}$.

35. $\frac{1+x}{1-x}-\frac{2+3x}{2-3x}=1+\frac{1+3x}{1-3x}$.

36. $\frac{1}{x-4}-\frac{1}{x-3}=\frac{1}{4}\left(\frac{1}{x-5}-\frac{1}{x-1}\right)$.

37. $\dfrac{x-1}{x-5}+\dfrac{x-5}{x-9}+\dfrac{x-9}{x-1}=3.$ **38.** $\dfrac{1}{x-3}-\dfrac{1}{x-5}-\dfrac{1}{x-7}+\dfrac{1}{x-9}=0.$

39. $\dfrac{3x-4\frac{1}{4}}{2x-3\frac{1}{8}}-\dfrac{7x+4}{8x-7}=\dfrac{5}{8}.$ **40.** $\dfrac{1}{x-3}-\dfrac{1}{x-4}=\dfrac{1}{x-6}-\dfrac{1}{x-7}.$

41. $\dfrac{5x-34}{x-7}+\dfrac{3x-26}{x-9}=\dfrac{5x-24}{x-5}+\dfrac{3x-32}{x-11}.$ **42.** $\dfrac{x-1}{x+1}+\dfrac{x+1}{x-2}+\dfrac{x-2}{x-1}=3.$

CHAPTER XXIII.

MISCELLANEOUS FACTORS FOR REVISION.

XXIII. a.

[Grouped in batches of 10.]

Resolve into their simplest factors:

1. $ax^2-bx.$ **2.** $x^2+11x+10.$ **3.** $3x^2-3.$ **4.** $2x^2-8x+6.$
5. $ax-bx+a^2-b^2.$ **6.** $1-2x-3x^2.$ **7.** $4a^3-4b^3.$
8. $18x^2+24x+6.$ **9.** $8x^2+14x-15.$ **10.** $x^3+2x^2-x-2.$

11. $20xy-15y^2.$ **12.** $ax^2-ab^2.$ **13.** $x^2-52x+51.$ **14.** $4(a^2-\frac{1}{4}).$
15. $x^3+ax^2+a^2x+a^3.$ **16.** $72-x-x^2.$ **17.** $(a+b)^2-a-b.$
18. $16x^2-50x-21.$ **19.** $a^2-b^2-c^2+2bc.$ **20.** $abx^2-4ax-3bx+12.$

21. $3-6x+3x^2.$ **22.** $27x^2-12x+1.$ **23.** $20a^2-45.$
24. $3ax+2by-2bx-3ay.$ **25.** $3a^3-81.$ **26.** $6+3x-2x^2-x^3.$
27. $35x^2+12x-32.$ **28.** $x^2y^2+1-x^2-y^2.$ **29.** $6-5x-2x^2+x^3.$
30. $a^3x^2+b^3y^2-a^3y^2-b^3x^2.$

31. $63ab-21bc-245b^2.$ **32.** $54x^2+15xy-y^2.$ **33.** $6x-ay-ax+6y.$
34. $3x^2-\frac{1}{3}.$ **35.** $27x^2-6x-8.$ **36.** $343x^2-7y^2.$
37. $x^2y^2-1-x^2+y^2.$ **38.** $(a-b)^3-a+b.$ **39.** $x^6-64y^6.$
40. $(a+b)^2-5a-5b+6.$

41. $p^3x^2-2p^2x+p.$ **42.** $x^2-25x+156.$ **43.** $x(x+8)+8(x+6).$
44. $33x^2+20xy-32y^2.$ **45.** $x^2+2ax-7bx-14ab.$ **46.** $(a+b)^3-(a-b)^3.$
47. $15x^2-2ab-5ax+6bx.$ **48.** $2x^6-128.$ **49.** $4x^3-7x-3.$
50. $(bx+ay)^2+(by-ax)^2-c^2(x^2+y^2).$

51. $x^2-16(x-4).$ **52.** $(a+\frac{1}{2})^2-(b+\frac{1}{2})^2.$ **53.** $x^2+14x-147.$
54. $3(a-b)^2-3a+3b.$ **55.** $12x^2-14ab+8ax-21bx.$ **56.** $x^3+3+2x^2-2x.$
57. $27x^2+210x-125.$ **58.** $x^2-3ay+3xy-a^2.$ **59.** $a^4-16(b-c)^4.$
60. $a(a-1)x^2+x-a(a+1).$

Resolve into their simplest factors

61. $a^2 + 2a + b^2 + 2b + 2ab$ 62. $35x^2 - 74xy - 24y^2$ 63. $3(x^2 - y^2) - 4x + 4y$

64. $a^4 - b^4$ 65. $x^4 + 2x^2y^3 + y^6$ 66. $16\left(x^2 - \dfrac{a^2}{16}\right)$

67. $32x^3 + 352x^2 + 320x$ 68. $(x+y)^2(x-y) - (x-y)^2(x+y)$

69. $4b^2c^2 - (a^2 - b^2 - c^2)^2$ 70. $(2a-b)^4 - (a-2b)^4$

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71. $5a^2 - a - 5b^2 + b$ 72. $39x^2 + 14x - 8$ 73. $16(x^4 - \frac{1}{16})$

74. $ax + by - ay - cx - bx + cy$ 75. $(x^2-2)^2 - x^2$

76. $(x+y)^2 - 13(x+y)a + 42a^2$ 77. $(3a-b)^4 - (a-3b)^4$

78. $a^2x + ac - abx - b^2y - bc + aby$ 79. $8(2x+y)^3 + (x-2y)^3$

80. $16x^4 + 4x^2y^2 + y^4$

REVISION PAPERS

XXIII. b.

1. Resolve the following into their simplest factors

(i) $ax^2 - a^3$ (ii) $x^2 - 2xy - 99y^2$

(iii) $75x^2 - 76x + 1$ (iv) $x^2 + xy - 5x - 5y$

2. Find the H.C.F. of $2x^2 - 5x - 3$ and $3x^3 - 81$.

3. Simplify $\dfrac{3}{x-1} - \dfrac{4}{x-2} + \dfrac{1}{x-3}$, and find a value of x which will make the expression equal to zero.

4. Multiply $x^2 - ax + bx - ab$ by $x^2 + ax - bx - ab$.

5. Using half an inch as x unit, and one tenth of an inch as y unit, plot the points given by the table below, and join them by an even curve.

x	-5	-4	-3	-2	-1	0	1	2	3	4	5
y	25	16	9	4	1	0	1	4	9	16	25

Read off from the figure, the values of x when $y = 7$ and 13, and the values of y when $x = 1.8$ and -2.4.

6. Solve the equation $\dfrac{x^2 - 2x + 4}{x-1} = \dfrac{x^2 - 5}{x+1}$.

7. A bicycles at the rate of 12 m. an hour, stopping for 6 minutes at the end of each hour. B starts 2 hours 24 minutes later on his motor car, and pursuing him catches him up 42 miles from the start without any stops. At what rate did B travel? Solve the problem graphically and algebraically.

XXIII. c.

1. Resolve the following into factors:

(i) $2x^2 - 8$ (ii) $2x^2 - 5x + 2$

(iii) $a^2 + 2ab + b^2 - c^2$ (iv) $x^2 - y^2 - 3x + 3y$

2. Simplify $\dfrac{(x^2-1)(x^2-4)}{(x^2+x-2)(x^2-x-2)}$.

3. Find the L.C.M. of $3a^3b-3a^2b^2$, $4ab^3-4a^2b^2$, $2a^3b^3$.

4. Simplify $[(x-1)^2+2(x-1)(2x-1)+(2x-1)^2]\div(3x-2)$.

5. Plot the points (10, 10), (15, 18), (30, 22), (39, 10). If the quadrilateral joining them represents a field, each square unit representing one-tenth of an acre, find the area of the field.

6. Solve the equations $\dfrac{1}{3x}-\dfrac{1}{4y}=\dfrac{11}{72}$, $\dfrac{1}{x}-\dfrac{1}{3y}=\dfrac{7}{18}$. Check your result.

7. A train does a journey without stoppages in 8 hours; if it had travelled 5 m. an hour faster, it would have done the journey in 6 hours 40 minutes. Find its slower speed.

XXIII. d.

1. Resolve into factors:

   (i) $2x^2+7x+3$. (ii) $a^2-b^2-2bx-x^2$.

   (iii) $c^2+ab-ac-bc$. (iv) $3-3b^3$.

2. Find the H.C.F. of $x^2-ax-bx+ab$, $x^2+cx-ax-ac$, and bx^2-a^2b.

3. Simplify $\dfrac{1}{x-y}-\dfrac{2x+y}{x^2-y^2}+\dfrac{x(x^2+y^2)}{x^4-y^4}$.

4. Draw the graph of $x+2y=8$, and from it write down all the positive integral solutions of the equation, not counting zero values.

5. Divide a^6-b^6 by a^2-ab+b^2.

6. Solve the equation $\dfrac{x^2-x-2}{x-2}+\dfrac{2x^2-x-1}{x-1}=\dfrac{4x^2+x-3}{x+1}$.

7. In an innings of a cricket eleven the team were accounted for in the following manner. Some were stumped, half as many again were caught, and half the wickets that fell were bowled. How many were stumped, caught, and bowled respectively?

XXIII. e.

1. Resolve into factors:

   (i) $x^2-28x-128$. (ii) $ax-2y-2x+ay$.

   (iii) x^3-5x^2+7x-3. (iv) $4+108a^3$.

2. Simplify $\dfrac{(a+b)^2-c^2}{(a-b)^2-c^2}\times\dfrac{(b+c)^2-a^2}{(c-b)^2-a^2}\div\dfrac{(a+b+c)^2}{c^2-(a-b)^2}$.

3. Find the L.C.M. of x^2-5x+6, x^2-x-2, x^2-2x-3.

4. A bicycles a journey of 36 miles in $5\frac{1}{2}$ hours, and B, starting $1\frac{1}{2}$ hours after him, arrives at the end of the journey 36 minutes before him. If they ride at uniform speeds, find graphically where B passes A. Calculate your result to the nearest tenth of a mile.

5. Divide $6x^4-5x^3+6x^2+17x+6$ by $6x^2+7x+2$.

6. Simplify $\dfrac{2x^2-5x+3}{2x-3}-\dfrac{3x^2+x-4}{x-1}+\dfrac{2(3x^2-13x-10)}{3x+2}$.

7. What value of x will make

   $(x+\frac{1}{2})^2-(x-\frac{3}{2})^2$ equal to $2x+3$.

XXIII. f.

1. Resolve into factors:

(i) $2x^2+9x-5$. (ii) $(2a+b)^2-(a+2b)^2$.

(iii) $a(b+c-d)+d(a-b-c)$. (iv) $x^3-x^2z-xy^2+y^2z$.

2. Find the H.C.F. of $c^2-(a-b)^2$, $(a+c)^2-b^2$, $(c-b)^2-a^2$.

3. Simplify $\dfrac{2}{1-x}-\dfrac{2}{2-x}+\dfrac{1}{(1-x)^2}-\dfrac{5}{(2-x)^2}$. Check your result by putting $x=3$.

4. Draw the graph of $2x+3y=21$, and from it write down all positive integral solutions, counting zero values as positive.

5. Solve the equations $\dfrac{5}{y}-\dfrac{2}{x}=1\tfrac{1}{8}$,

$\dfrac{36}{x}-\dfrac{24}{y}=1$. Check your results.

6. By doing a journey at the rate of $12\tfrac{1}{2}$ miles an hour a bicyclist completes it in 3 minutes less time than if he had travelled at 12 miles an hour. Find the length of the journey.

7. Solve the equation $\dfrac{x+5}{x+4}-\dfrac{x+7}{x+6}=\dfrac{x+10}{x+9}-\dfrac{x+12}{x+11}$. Test your answer.

XXIII. g.

1. Resolve into factors:

(i) $12x^2+7x-12$. (ii) $4a^2+b^2-c^2-d^2+4ab+2cd$.

(iii) $x^3-2-x+2x^2$. (iv) $x^2y^2-x^2-y^2+1$.

2. Simplify $\dfrac{x^4+x^2+1}{x^4-4}\times\dfrac{x^2-2}{x^3-1}\div\dfrac{x^3+1}{x^2+2}$.

3. Find the L.C.M. of $3(x^4-x^2y^2)$, $6(x^2y^2+y^4)$, $9(x^3-x^2y+xy^2-y^3)$.

4. The majority against a certain motion is equal to $6\tfrac{2}{3}$ per cent. of the total number voting. If 12 of those who voted against the motion had voted for it, the motion would have been carried by a single vote. Find the numbers voting on each side.

5. Divide $x^3-b(4a+b)x+(a+2b)(a^2+3b^2)$ by $x+a+2b$.

6. Solve the equation $\dfrac{2x+3}{x+1}-\dfrac{2x+9}{x+4}=\dfrac{3x+7}{x+2}-\dfrac{3x+16}{x+5}$. Test your answer.

7. A man travels at the rate of x feet per minute.

How long does he take to do a mile?

How many yards does he travel in an hour?

How many miles does he travel in y hours?

XXIII. h.

1. Simplify $\left(x+\dfrac{1}{x}\right)^3-\left(x-\dfrac{1}{x}\right)^3$.

2. Solve the equation $\dfrac{2x^2+5x+4}{x+2}=\dfrac{4x^2+8x+6}{2x+3}$. Test your solution.

3. Plot the points (0, 0), (1, 1), (4, 2), (9, 3), (16, 4), (25, 5), (1, −1), (4, −2), (9, −3), (16, −4), (25, −5), using one-tenth of an inch as x unit, and half an inch as y unit. Join the points by an even curve. Estimate the corresponding y values on the curve when $x=11$, and when $x=23$.

4. Simplify $\frac{a^2-b^2}{a^2}\times\left(1+\frac{2b}{a-b}\right)^2\div\frac{(a+b)^3}{a^3-a^2b}$.

5. A fraction is such that its denominator exceeds its numerator by 2; also if the numerator is diminished by unity and the denominator increased by unity, the fraction becomes equal to $\frac{1}{2}$. Find the fraction.

6. Solve the equations $\frac{x}{y}-2x=2\frac{1}{2}$,

$\frac{x}{y}+2x+5\frac{1}{2}=0$. Test your solution.

7. What is the interest on

(i) £300 for 1 year at x per cent. per annum?

(ii) .. .. 4 years ...., simple interest?

(iii) £a for 1 year ...., ...?

(iv) .... . y years .. .. , ... ?

XXIII. k.

1. Divide $x^2+1+\frac{1}{x^2}$ by $x-1+\frac{1}{x}$.

2. Solve the equation $\frac{4}{5x-1}-\frac{17}{25x^2-1}=\frac{3}{5x+1}$. Test your solution

3. From the equation $\frac{3}{y-5}+\frac{4}{2-x}=\frac{14}{(x-2)(y-5)}$, find the value of $\frac{x}{y}$.

4. Simplify $\left(1-\frac{2y}{x}+\frac{y^2}{x^2}\right)\times\frac{x+y}{\frac{x}{y}-\frac{y}{x}}\div\left(\frac{x}{y}-\frac{y^2}{x^2}\right)$.

5. At what time (to the nearest minute) do the hands of a clock point in the same direction between 4 and 5 o'clock?

6. Solve the equations $xy+4x=7$,

$xy-3x=14$. Test your solution.

7. In the equation $y=2x-x^2$, find the corresponding values of y to all integral values of x from −3 to 5. Tabulate your work. Using half an inch as x unit, and one tenth of an inch as y unit, plot the points, and join them by an even curve.

XXIII. l.

1. Divide $(x^2-y^2)^2-(x^2-3xy+2y^2)^2$ by $(x-y)^2$.

2. Solve the equation $\frac{3x^2+14x+7}{x+4}=\frac{9x^2-5}{3x-2}$. Test your solution.

3. Simplify $\frac{a^2+b^2-c^2+2ab}{a^2+b^2-c^2-2ab}\div\frac{a+b+c}{a-b+c}$.

4. Find two numbers whose difference is 27, such that the larger divided by the smaller gives a quotient 7 and a remainder 3.

5. Find values of a and b which will satisfy both the equations

$$\frac{a}{x}-\frac{b}{y}=7,\quad \frac{2a}{x}-\frac{3b}{y}=2,\ \text{when } x=\tfrac{1}{2} \text{ and } y=\tfrac{1}{3}.$$

6. Solve the equations

$$3x+4y+14=0,$$
$$5x-2y+6=0.$$

Deduce the solution of the equations

$$\frac{3}{x}+\frac{4}{y}+14=0,$$
$$\frac{5}{x}-\frac{2}{y}+6=0.$$

7. If $2x-3y-1=0$, and $xy-3x+2=0$, prove that $3y^2-8y+1=0$.

XXIII. m.

1. Divide $(a^2+2ab-3b^2)^2-(a^2-4ab+3b^2)^2$ by $(a-b)^2$.

2. Solve the equation $\dfrac{3}{2x+3}-\dfrac{1}{2-x}=\dfrac{19}{2(2x+3)(x-2)}$. Test your solution.

3. From the equation $\dfrac{7}{y-4}-\dfrac{3}{x-2}+\dfrac{2}{(x-2)(y-4)}=0$, find the value of $\dfrac{x}{y}$.

4. Simplify $\dfrac{4x^4+8x^2+4}{(x^2-x+1)^2}\times\dfrac{x^4+x^2+1}{x^4+1}\div\dfrac{(x^4+x^2)^2}{x^3+1}$.

5. At what time (to the nearest minute) do the hands of a clock point in opposite directions between 4 and 5 o'clock?

6. Try to solve the equation $\dfrac{1}{x+4}=\dfrac{2}{2x-7}+\dfrac{15}{(7-2x)(4+x)}$. What conclusion do you draw?

7. A horse is bought for £85, and sold at a gain of x per cent. What is the selling price?

By selling a horse for £92, a profit of x per cent. is made: what was the original price of the horse?

CHAPTER XXIV.

SQUARE ROOT.

124. Every quantity has two square roots, equal in value but opposite in sign.

E.g. the square root of 4 is $+2$ or -2,

for $(+2)^2=4$, and $(-2)^2=4$.

$\therefore\ \sqrt{4}=2$ or -2,

or, as it is written more shortly, $\sqrt{4}=\pm 2$.

At present we will only deal with the positive root.

A square is always positive, for by the rule of signs

$$a \times a = a^2,$$
$$(-a) \times (-a) = a^2;$$

i.e. whether a quantity is positive or negative, its square is positive.

Hence we see that a negative quantity has no square root. The square root of a negative quantity however has an interpretation, but this hardly comes into the province of Elementary Algebra.

The square roots of simple algebraical expressions can be seen by inspection.

$$\sqrt{(a^4b^2)} = a^2b.$$
$$\sqrt{x^2y^4z^6} = xy^2z^3.$$
$$\sqrt{16a^4} = 4a^2.$$
$$\sqrt{\frac{81b^4}{x^2}} = \frac{9b^2}{x}.$$

Examples. XXIV. a.

Write down, or read off, the positive square roots of the following:

1. x^8.	2. a^{10}.	3. y^{16}.	4. x^6y^4.
5. a^2b^4.	6. x^8y^6.	7. $4a^2b^2$.	8. $16a^4b^2$.
9. $49x^4y^6z^8$.	10. $\frac{4a^2}{b^2}$.	11. $\frac{9z^4}{y^6}$.	12. $\frac{81a^4b^6}{c^8}$.
13. ·01.	14. ·25.	15. ·64.	16. $\frac{1}{\cdot 0001}$.
17. $\frac{1}{\cdot 16}$.	18. $\frac{\cdot 49}{\cdot 36}$.	19. $\cdot 01b^4c^2$.	20. $\frac{\cdot 16a^2}{4b^4}$.
21. $1\cdot 21a^6c^{10}$.	22. $\frac{16}{49}x^{12}y^{16}$.	23. $\frac{a^4}{\cdot 81b^2}$.	24. $\frac{\cdot 0064x^4}{\cdot 0001y^{12}}$.
25. $9(a-b)^2$.	26. $\frac{121}{9}(2x+y)^2$.	27. $\cdot 01(10x+10y)^2$.	

125. The square of a simple expression is also a simple expression.

E.g.
$$(4a^2b^2)^2 = 16a^4b^4.$$

We know also that the square of a binomial expression is a trinomial expression

E.g.
$$(x+2)^2 = x^2 + 4x + 4,$$
$$(2x+3)^2 = 4x^2 + 12x + 9.$$

Thus we see that a binomial expression has no square root.

126. The square root of a trinomial expression which is a square can usually be determined by inspection.

$$(a \pm b)^2 = a^2 \pm 2ab + b^2.$$

Hence all trinomials which are perfect squares must be of the form $$a^2 + 2ab + b^2.$$

Thus $$4x^2 + 12xy + 9y^2 = (2x)^2 + 2(2x)(3y) + (3y)^2.$$

$$\therefore \sqrt{4x^2 + 12xy + 9y^2} = 2x + 3y.$$

$$\sqrt{4x^2 - 12xy + 9y^2} = 2x - 3y.$$

The form of the square of a binomial $(a^2 \pm 2ab + b^2)$ is of great importance.

Consider the expression

$$x^2 + pax + a^2.$$

By comparing this with the above we see that if it has a square root, that root must be $x + a$.

But $$(x + a)^2 = x^2 + 2ax + a^2;$$

$\therefore$ if $x^2 + pax + a^2$ is a perfect square,

$$p \text{ must be equal to } 2.$$

Examples. XXIV. b.

Determine the square roots of the following expressions:

1. $x^2 + 2xy + y^2$. **2.** $x^2 - 2xy + y^2$. **3.** $a^2 + 4ab + 4b^2$.

4. $4a^2 - 4ab + b^2$. **5.** $x^2 - 6x + 9$. **6.** $1 - 4x + 4x^2$.

7. $25a^2 - 30ab + 9b^2$. **8.** $49x^2 - 14xy + y^2$. **9.** $4a^2 - 28ab + 49b^2$.

10. $9x^2 + 24xy + 16y^2$. **11.** $121a^2 - 44ab + 4b^2$. **12.** $1 - 2x^3 + x^6$.

13. $169a^2 + 52ab + 4b^2$. **14.** $81a^2 - 18ab + b^2$. **15.** $25x^2 - 70xy + 49y^2$.

16. $a^4 - 2a^2b^2 + b^4$. **17.** $4a^4 + 4a^2b^2 + b^4$. **18.** $x^4y^2 - 2x^2y + 1$.

19. $\frac{x^2}{9} - \frac{2x}{3} + 1$. **20.** $a^4 + 4a^2b^2 + 4b^4$. **21.** $x^2 - x + \frac{1}{4}$.

22. $\frac{a^2}{4} - ab + b^2$. **23.** $\frac{x^2}{y^2} - 2 + \frac{y^2}{x^2}$. **24.** $x^2 - 3xy + \frac{9y^2}{4}$.

25. $x^4 + \frac{1}{x^4} + 2$. **26.** $a^2 - 5a + \frac{25}{4}$. **27.** $(x+y)^2 + 2(x+y) + 1$.

28. $(a+b)^2 - 2(a^2 - b^2) + (a-b)^2$. **29.** $(x-y)^2 - 4(x-y) + 4$.

30. $9(a+b)^2 + 6(a+b) + 1$. **31.** $(a+b)^2 + 2(a+b)(c+d) + (c+d)^2$.

32. $(a+b)^2 + 2a(a+b) + a^2$. **33.** $\left(\frac{a}{b}+1\right)^2 - 2\left(\frac{a}{b}+1\right) + 1$.

34. $16(x-y)^2 - 8(x-y) + 1$. **35.** $(a+2b)^2 + (a+2b) + \frac{1}{4}$.

36. $(a+b)^2 - 2a(a+b) + a^2$. **37.** $\left(\frac{a}{b}-1\right)^2 - 2\left(\frac{a}{b}-1\right) + 1$.

38. $16(x+y)^2-24(x^2-y^2)+9(x-y)^2$.

39. $\frac{a^6}{x^6}-2+\frac{x^6}{a^6}$.

40. $\frac{4a^4}{x^4}-4+\frac{x^4}{a^4}$.

41. $\frac{x^8}{4a^8}+2+\frac{4a^8}{x^8}$.

42. $\frac{(a+b)^2}{9}-\frac{(a+b)(x+y)}{3}+\frac{(x+y)^2}{4}$.

What must be added to the following expressions to make them complete squares?

43. a^2+b^2.　**44.** x^2-4x.　**45.** $9+x^2$.

46. $4x^2+25y^2$.　**47.** $(a+b)^2+2(a+b)$.

48. Determine the value of p if $x^2-4px+16$ is a perfect square.

49. For what value of a will x^2-2x+a be a perfect square?

50. What value of p will make $x^2+6pxy+q^2y^2$ a perfect square?

127. *To find the square root of any compound expression.*

The method depends upon the fact that the square of $a+b$ is $a^2+2ab+b^2$, which may be written in the form

$$a^2+b(2a+b)\ldots\ldots\ldots\ldots\ldots(\text{i})$$

Let us take an easy example.

The first term in the square root of $36x^2-84xy+49y^2$ is evidently $6x$.

$$\begin{array}{l} 36x^2-84xy+49y^2\,(6x \\ \underline{36x^2} \\ \quad\;\; -84xy+49y^2 \end{array}$$

Subtracting its square, *i.e.* $36x^2$, from the given expression, the remainder is $-84xy+49y^2$, which may be written

$$-7y(2\times 6x-7y).$$

Comparing this with (i), we see that in this case a is $6x$, and therefore b is $-7y$.

Hence we have the following rule.

Having obtained the first term, (6x), double it, (12x), and divide the first term (−84xy) of the remainder by it. The quotient (−7y) is the second term of the square root.

The full work is best arranged as below:

$$\begin{array}{rl} & 36x^2-84xy+49y^2\,(\,6x-7y \\ & \underline{36x^2\qquad\qquad\qquad} \\ & \qquad -84xy+49y^2 \\ (12x-7y)\times(-7y)= & \qquad \underline{-84xy+49y^2} \end{array}$$

Explanation. Having obtained the first term of the square root, $6x$, we double it, $12x$, and divide it into $-84xy$, the first

term of the remainder when $(6x)^2$ is subtracted. The quotient $(-7y)$ is the second term of the answer.

Add $-7y$ to $12x$ and multiply the result by $-7y$, placing the result $-84xy+49y^2$ under the remainder.

If the student carefully compares the following with the expression $a^2+b(2a+b)$, he will see the reasons for the different steps.

$$\begin{array}{rl}
 & a^2+2ab+b^2\,(\,a \\
 & \underline{a^2\qquad\qquad\quad} \\
 & \qquad 2ab+b^2 \\
(2a+b)\times b = & \qquad \underline{2ab+b^2}
\end{array}$$

128. *Find the square root of*

$$25x^4-30px^3+49p^2x^2-24p^3x+16p^4.$$

$$\begin{array}{rl}
 & 25x^4-30px^3+49p^2x^2-24p^3x+16p^4\,(\,5x^2-3px \\
 & \underline{25x^4\qquad\qquad\qquad\quad} \\
 & \quad -30px^3+49p^2x^2 \\
(10x^2-3px)\times(-3px) = & \quad \underline{-30px^3+9p^2x^2} \\
 & \qquad\qquad\quad 40p^2x^2-24p^3x+16p^4
\end{array}$$

Thus far the work is exactly similar to that in the previous examples, the reasons being the same.

Thinking once more of the expression $a^2+b(2a+b)$, we see that if the given expression has a square root, the remainder $40p^2x^2-24p^3x+16p^4$ must be of the form $b(2a+b)$, remembering that now a is $5x^2-3px$.

We therefore *repeat the process of the first step.*

Double $5x^2-3px$, obtaining $10x^2-6px$.

$40p^2x^2 \div 10x^2 = 4p^2$ gives us the next term of the answer.

Add this to $10x^2-6px$, obtaining $10x^2-6px+4p^2$; multiply this by $4p^2$, and place the result under the remainder.

The example is worked out in full below:

$$\begin{array}{rl}
 & 25x^4-30px^3+49p^2x^2-24p^3x+16p^4\,(\,5x^2-3px \\
 & \underline{25x^4\qquad\qquad\qquad\quad} \qquad\qquad\qquad\qquad\quad +4p^2 \\
 & \quad -30px^3+49p^2x^2 \\
(10x^2-3px)\times(-3px) = & \quad \underline{-30px^3+9p^2x^2} \\
 & \qquad\qquad\quad 40p^2x^2-24p^3x+16p^4 \\
(10x^2-6px+4p^2)\times 4p^2 = & \qquad\qquad\quad \underline{40p^2x^2-24p^3x+16p^4}
\end{array}$$

$5x^2-3px+4p^2$ is the reqd. sq. root.

129. The square root of a compound expression can often be seen by re-arrangement and inspection.

$$\begin{aligned} &x^4 - 2x^3 - x^2 + 2x + 1 \\ &\qquad = x^4 - 2x^3 - 2x^2 + (x^2 + 2x + 1) \\ &\qquad = x^4 - 2x^2(x+1) + (x+1)^2 \quad [a^2 - 2ab + b^2] \\ &\qquad = [x^2 - (x+1)]^2 ; \end{aligned}$$

$$\therefore \ \sqrt{x^4 - 2x^3 - x^2 + 2x + 1} = x^2 - x - 1.$$

$$\begin{aligned} &a^2 + b^2 + c^2 - 2bc - 2ac + 2ab \\ &\qquad = a^2 + 2a(b-c) + b^2 + c^2 - 2bc \end{aligned}$$

(arranging in descending powers of a)

$$\begin{aligned} &\qquad = a^2 + 2a(b-c) + (b-c)^2 \\ &\qquad = (a+b-c)^2 ; \end{aligned}$$

$$\therefore \ \sqrt{a^2 + b^2 + c^2 - 2bc - 2ac + 2ab} = a + b - c.$$

Find the square root of

$$\frac{4x^4}{25} + \frac{1}{9x^4} - \frac{4x^2}{5} - \frac{2}{3x^2} + \frac{19}{15}.$$

Arrange the expression in *descending* powers of x.

$$\begin{array}{ll} & \dfrac{4x^4}{25} - \dfrac{4x^2}{5} + \dfrac{19}{15} - \dfrac{2}{3x^2} + \dfrac{1}{9x^4}\left(\dfrac{2x^2}{5} - 1 + \dfrac{1}{3x^2}\right. \\ & \dfrac{4x^4}{25} \\ \hline & \qquad -\dfrac{4x^2}{5} + \dfrac{19}{15} \\ \left(\dfrac{4x^2}{5} - 1\right) \times (-1) & \qquad -\dfrac{4x^2}{5} + 1 \\ \hline & \qquad\qquad \dfrac{4}{15} - \dfrac{2}{3x^2} + \dfrac{1}{9x^4} \\ \left(\dfrac{4x^2}{5} - 2 + \dfrac{1}{3x^2}\right) \times \dfrac{1}{3x^2} & \qquad\qquad \dfrac{4}{15} - \dfrac{2}{3x^2} + \dfrac{1}{9x^4} \\ \hline \end{array}$$

Examples. XXIV.

Find the square roots of the following expressions :

1. $x^4 + 2x^3 + 3x^2 + 2x + 1.$
2. $4x^4 + 4x^3 + 5x^2 + 2x + 1.$
3. $x^4 - 2x^3 + 5x^2 - 4x + 4.$
4. $a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4.$
5. $9x^4 - 12x^3 + 34x^2 - 20x + 25.$
6. $4x^2 + 25y^2 + 16z^2 - 20xy - 40yz + 16zx.$
7. $16x^6 + 6x^3 + 17x^4 + x^2 + 24x^5.$
8. $12a^3x - 26a^2x^2 + 25x^4 + 9a^4 - 20ax^3.$

Find the square roots of the following expressions :

9. $x^4 - 6x^2 + 11 - \frac{6}{x^2} + \frac{1}{x^4}$.

10. $a^2 + b^2 + c^2 - 2ab + 2bc - 2ca$.

11. $x^6 - 6x^4 + 49 + 42x - 14x^3 + 9x^2$.

12. $9x^4 - 12x^3y + 34x^2y^2 - 20xy^3 + 25y^4$.

13. $a^2 + 4b^2 + 9c^2 - 4ab - 12bc + 6ca$.

14. $9a^4 + 49b^4 + 121c^4 - 42a^2b^2 + 154b^2c^2 - 66a^2c^2$.

15. $4a^2b^2 + 9b^2c^2 + c^2a^2 - 4a^2bc - 12ab^2c + 6abc^2$.

16. $4x^2 + 9y^2 + 25z^2 - 12xy + 20xz - 30yz$

17. $49x^4 + 109x^2y^2 + 36y^4 - 70x^3y - 60xy^3$.

18. $x^6 - 4x^3 + 2 + \frac{4}{x^3} + \frac{1}{x^6}$.

19. $4x^4 + 9y^4 + 49z^4 - 12x^2y^2 - 42y^2z^2 + 28x^2z^2$.

20. $\frac{x^2}{y^2} + \frac{y^2}{x^2} + 3 - 2\left(\frac{x}{y} + \frac{y}{x}\right)$.

21. $\frac{a^4}{4} - a^3 + 2a + 1$.

22. $\frac{a^4}{9} + \frac{2a^3}{3} + \frac{4a^2}{3} + a + \frac{1}{4}$.

23. $\frac{9a^4}{25} + \frac{4a^3}{5} + \frac{74a^2}{45} + \frac{4a}{3} + 1$.

24. $\frac{a^4}{9} - \frac{a^3}{3} + \frac{11a^2}{12} - a + 1$.

25. $x^6 - x^5 + \frac{x^4}{4} + \frac{2x^3}{3} - \frac{x^2}{3} + \frac{1}{9}$.

26. $\frac{x^4}{4} - 3x^3 + \frac{28x^2}{3} - 2x + \frac{1}{9}$.

27. $\frac{x^4}{9} + \frac{a^4}{4} - \frac{4x^3}{3} + \frac{ax^2}{3} + 4x^2 - 2ax$.

28. $9x^4 + \frac{64}{x^4} + 24x^2 - \frac{64}{x^2} - 32$.

29. $\frac{4x^2}{y^2} + \frac{9y^2}{4x^2} - \frac{x}{y} + \frac{3y}{4x} - \frac{95}{16}$.

30. $\frac{16}{25} - \frac{3a^3}{2} + a^4 - \frac{6a}{5} + \frac{173a^2}{80}$.

SQUARE ROOT OF NUMERICAL QUANTITIES.

130. First study carefully the following example worked according to the algebraic method.

Example. Find the square root of 99225.

$$\begin{array}{rl}
 & 99225 = 9 \cdot 10^4 + 9 \cdot 10^3 + 2 \cdot 10^2 + 2 \cdot 10 + 5\,(\,3 \cdot 10^2 + 1 \cdot 10 + 5 = 315 \\
 & \underline{9 \cdot 10^4} \\
 & \quad 9 \cdot 10^3 + 2 \cdot 10^2 \\
(6 \cdot 10^2 + 1 \cdot 10) \times (1 \cdot 10) = & \underline{6 \cdot 10^3 + 1 \cdot 10^2} \\
 & 3 \cdot 10^3 + 1 \cdot 10^2 + 2 \cdot 10 + 5 \\
(6 \cdot 10^2 + 2 \cdot 10 + 5) \times (5) = & \underline{3 \cdot 10^3 + 1 \cdot 10^2 + 2 \cdot 10 + 5}
\end{array}$$

Below we give the same example in arithmetical form, omitting superfluous powers of 10.

$$\begin{array}{rl}
 & 9,92,25\,.\,(\,315 \\
 & \underline{9} \\
 & 92 \\
(60 + 1) \times 1 = 61 & \underline{61} \\
 & 3125 \\
(620 + 5) \times 5 = 625 \times 5 = & \underline{3125}
\end{array}$$

131. The following are very useful and should be learnt by heart:

$$13^2 = 169, \quad 17^2 = 289,$$
$$14^2 = 196, \quad 18^2 = 4 \times 81 = 324,$$
$$15^2 = 9 \times 25 = 225, \quad 19^2 = 361,$$
$$16^2 = 4 \times 64 = 256, \quad 21^2 = 9 \times 49 = 441.$$

132. The square roots of numerical quantities can often be best found by using factors.

$$1764 = 4 \times 441 = 4 \times 9 \times 49; \quad \therefore \sqrt{1764} = 2 \times 3 \times 7 = 42.$$
$$53361 = 9 \times 5929 = 9 \times 7 \times 847 = 9 \times 7 \times 7 \times 121 = 3^2 \times 7^2 \times 11^2;$$
$$\therefore \sqrt{53361} = 3 \times 7 \times 11 = 231.$$

Examples. XXIV. d.

Find the square root of

1. 1,764.	**2.** 18,225.	**3.** 16,900.	**4.** 2,704.
5. 34,969.	**6.** 390,625.	**7.** 213,444.	**8.** 7,056.
9. 15,876.	**10.** 4,020,025.	**11.** 9,006,001.	**12.** 3,892,729.
13. 5,499,025.	**14.** 408,120,804.	**15.** 1,825,201.	**16.** 12,173,121.

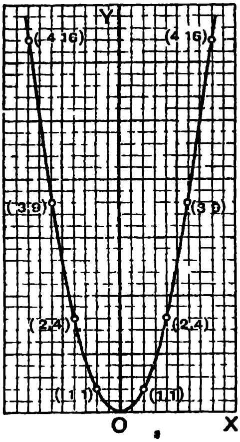
THE DETERMINATION OF THE SQUARE ROOTS OF NUMBERS BY GRAPHICAL METHODS

133. The student must first familiarize himself with the graph of the equation $y=x^2$

Trace the graph of $y=x^2$

When

$x=0$	±1	±2	±3	±4	±5	
$y=0$	1	4	9	16	25	



Joining these points, we have the graph reqd , which we see is a curve

For every value of y there are two equal and opposite values of x

∴ the curve is symmetrical about the axis of y

Moreover, as x increases indefinitely, y also increases indefinitely

∴ the parts of the curve on either side of OY meet only at the origin

Such a curve is called a **parabola**

N.B.—In the above we have taken twice the length of the side of a square to denote unity.

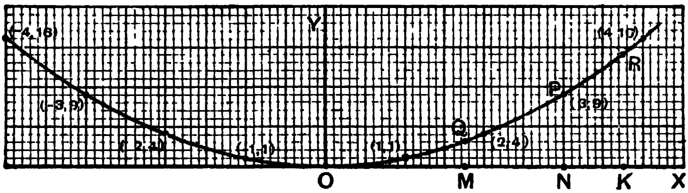
We observe that when x is greater than unity, the y value increases much more rapidly than the x value. This is well seen from the table of corresponding values of x and y below.

When

$x=$	5	6	7	8	9	10	11	...
$y=$	25	36	49	64	81	100	121	..

134. A better curve for working purposes will be obtained if we take 10 times the side of a square to denote unity for the abscissae, and one side of a square to denote unity for the ordinates.

Employing these units, we obtain the curve shown below.



Thus at P, the abscissa ON = 30 times the side of a sq. = 3 units

and the ordinate PN = 9 times the side of a sq. = 9 units.

The effect of using different units for the x and y values in this way, is the same as uniformly stretching the paper in a direction parallel to the axis of x. If we took the larger unit for the y values, it would be the equivalent of stretching the paper parallel to the axis of y.

It will sometimes be found convenient to take the x unit still larger.

In connection with square roots, the important thing to observe is that since $\mathbf{y = x^2}$, or $\mathbf{x = \sqrt{y}}$, **for every point on the curve, the abscissa of any point on it is the square root of the corresponding ordinate.**

In the curve shown above take the point Q when the ordinate is 3 and draw the ordinate QM.

Now at every pt. on the curve $y=x^2$;

$\therefore$ at Q $3=\text{OM}^2$, for there $y=3$ and $x=\text{OM}$;

$\therefore$ $\text{OM}=\sqrt{3}$.

But from the figure we see that OM lies between 1·7 and 1·8, and somewhat nearer 1·7 than 1·8;

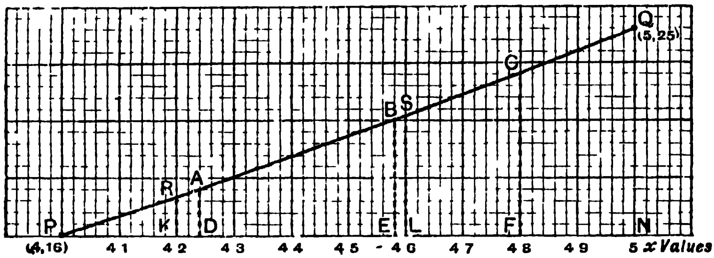
$\therefore$ $\sqrt{3}=1\cdot7$ correct to one decimal place.

Again take the pt. R where RK, the ordinate, $=14$.

$14=\text{OK}^2$;

$\therefore$ $\sqrt{14}=\text{OK}=3\cdot7$ correct to one decimal place.

135. *Construct a graph from which the square roots (correct to two decimal places) of numbers between* 16 *and* 25 *may be read off.*



We must draw the graph of $y=x^2$, and use a large unit for x values, for x has to be determined accurately to two decimal places.

We shall only need to draw that part of the curve where x lies between 4 and 5.

Take 50 sides of squares to represent unity in the x values, and 2 sides of squares to represent unity in the y values

In the curve $y=x^2$, when $x=4$, $y=16$,

and when $x=5$, $y=25$.

Let P be the pt. (4, 16) and Q the pt. (5, 25) so that PN in the figure representing unity is equal to 50 sides of squares, and QN representing 9 is equal to 18 sides of squares.

($N.B.$—QN is the difference of the ordinates of P and Q, and therefore $= 25 - 16 = 9$ units)

When $x = 4{\cdot}2$, $y = x^2 = (4{\cdot}2)^2 = 17{\cdot}64$,

$$17{\cdot}64 - 16 = 1{\cdot}64 \text{ units} = 3{\cdot}28 \text{ sides of sqs.}$$

Hence estimating the value of $\cdot 28$, R in the fig. is the pt. $(4{\cdot}2, (4{\cdot}2)^2)$.

(RK in the fig. = the diff. of the ordinates of R and P

$= 17{\cdot}64 - 16 = 1{\cdot}64$ units $= 3{\cdot}28$ sides of sqs.)

Again, when $x = 4{\cdot}6$, $y = x^2 = (4{\cdot}6)^2 = 21{\cdot}16$;

$\therefore$ estimating the value of $\cdot 16$, S in the fig. is the pt. $(4{\cdot}6, (4{\cdot}6)^2)$.

(Here again, SL = the diff. of the ordinates of S and P

$= 21{\cdot}16 - 16 = 5{\cdot}16$ units $= 10{\cdot}32$ sides of sqs.)

The curve through the pts. P, R, S, Q is evidently so nearly a str. line that we need find no more pts. on the curve.

Join the pts. P, R, S, Q by the continuous curve as shown in the figure.

To find $\sqrt{18}$ from this graph we must take the pt. whose ordinate is 18, *i.e.* the pt. A. ($N.B.$—AD $= 18 - 16 = 2$ units $= 4$ sides of a sq.)

From the fig. we see that the abscissa of this pt. is $4 + $ PD, which is equal to $4{\cdot}24$;

$$\therefore \sqrt{18} = 4{\cdot}24.$$

To find $\sqrt{21}$, we must take the pt. whose ordinate is 21, *i.e.* the pt. B. ($N.B.$—BE $= 21 - 16 = 5$ units $= 10$ sides of a sq.)

From the graph the abscissa of this point $= 4 + $ PE $= 4{\cdot}58$;

$$\therefore \sqrt{21} = 4{\cdot}58.$$

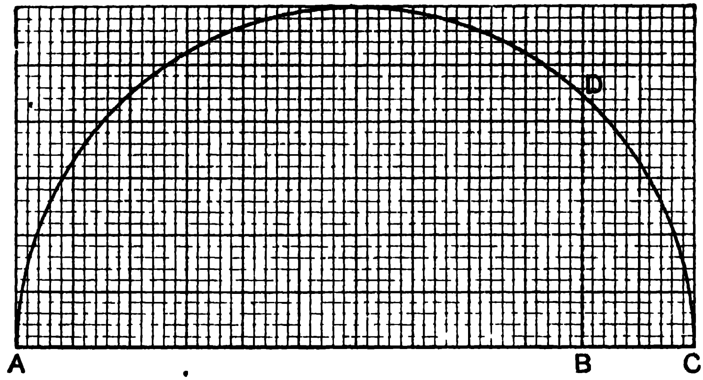
To find $\sqrt{23}$, we must take the pt. whose ordinate is 23, *i.e.* the pt. C;

$$\therefore \sqrt{23} = 4 + \text{PF} = 4{\cdot}80.$$

The roots of other numbers between 16 and 25 can be read off in the same way.

136. The following geometrical methods may be used for determining the values of square roots in simple cases.

Example. To find the value of $\sqrt{5}$.



First Method. Take AB 5 units long, and produce it to C making BC equal to one unit. On AC as diameter describe the circle ADC. At B draw BD perp. to AC, meeting the circle at D.

From geometry we know that

$$DB^2 = AB \,.\, BC = 5;$$

$$\therefore\ DB = \sqrt{5}.$$

From the diagram $\sqrt{5} \doteq 2 \cdot 24$ approx.

(If squared paper is not used, DB must be measured.)

Second Method. On AB, 5 in. long, as diameter describe a circle.

In AB take a pt. D 1 in. from A, and draw DC perp. to AB to meet the circle at C. Join AC. With centre A and radius AC describe a circle cutting AB at E.

By geometry $AC^2 = AD \,.\, AB = 5;$

$$\therefore\ AC = \sqrt{5};$$

$$\therefore\ AE = AC = \sqrt{5},$$

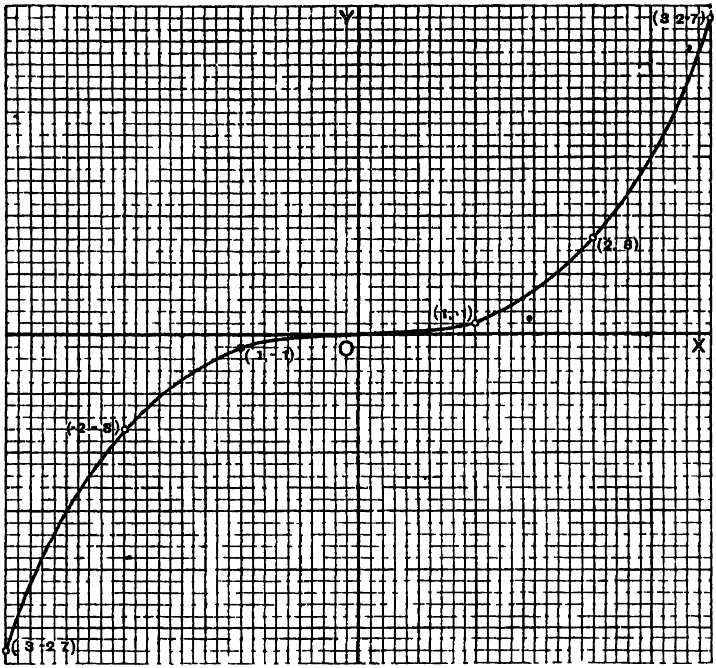
and if squared paper is used we can read off the value of $\sqrt{5}$ from the diagram.

Pythagoras' Theorem, which proves that the square on the hypotenuse of a right angled triangle is equal to the sum of the squares on its sides, may be sometimes used with advantage.

Thus to find $\sqrt{10}$, $10 = 1^2 + 3^2$, draw AB 3 units long, AC 1 unit long at rt. angles to AB. Join BC. $BC = \sqrt{10}$ units long.

CUBE ROOT BY GRAPHICAL METHOD.

***137.** *Draw the graph of $y = x^3$*



Use for the y values a unit one tenth of that for the x values

When

$x = 1$	2	3	4	5		inches
$y = 1$	8	27	64	125		tenths of an inch.

$x =$ 1	-2	-3		inches
$y =$ 1	8	-27		tenths of an i

Plot these points and we have the graph reqd

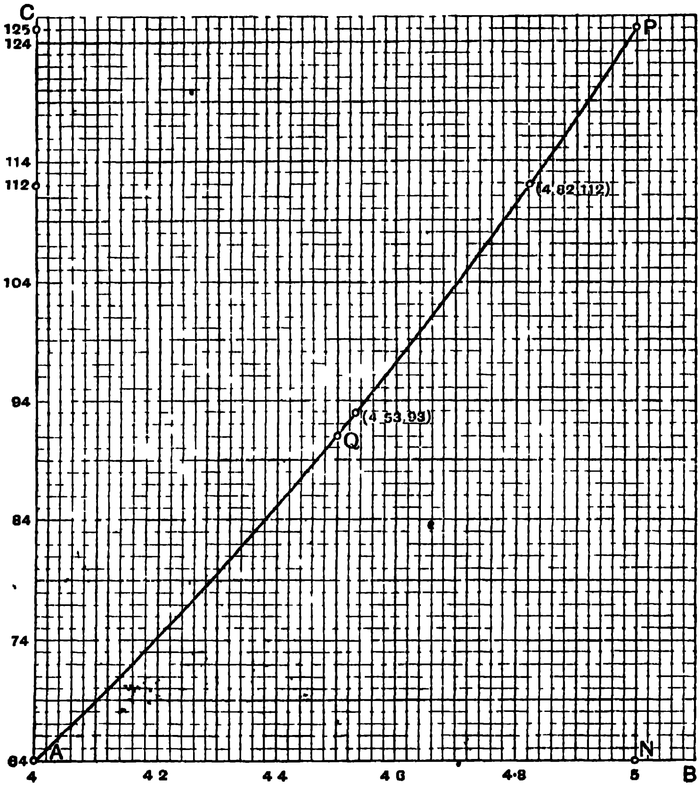
We see that the curve lies entirely in the first and third quadrants, and that the parts of the curve in those quadrants are similar

For values of x greater than 1 or less than -1, as the numerical value of x increases, that of y increases much more rapidly; but for values of x between 1 and -1 the reverse happens. This shows that the axis of x is a tangent to the curve at the origin.

As x varies continuously from $-\infty$ through 0 to $+\infty$, y also varies continuously from $-\infty$ through 0 to $+\infty$.

From this graph we can read off cube roots and cubes of numbers.

***138.** *To construct a graph from which the cube root of any number between* 64 *and* 125 *may be written down, correct to two decimal places.*



Take a piece of squared paper ruled in inches and tenths of an inch

Let the pt. A denote the pt. whose co-ors. are (4, 64).

In the horizontal line AB take 1 in. to represent ·2, so that AN (5 in. long) represents unity.

In the vertical line AC take an inch to represent 10.

On the paper plot the point (5, 125) P.

$(4{\cdot}5)^3 = 91{\cdot}125$. $\therefore$ plot the pt. (4·5, 91·125) Q, estimating the value of ·125.

Join the pts. A, Q, P by an even curve.

This curve will be seen to be part of the graph of $y = x^3$.

$\therefore$ we can read from it the values of the cube roots of numbers between 64 and 125.

E.g. $\sqrt[3]{112} = 4{\cdot}82$, $\sqrt[3]{93} = 4{\cdot}53$.

NOTE.—Great accuracy can be obtained in the above if a few more points are plotted; *e.g.* $[(4{\cdot}2), (4{\cdot}2)^3]$, $[(4{\cdot}8), (4{\cdot}8)^3]$.

Examples. XXIV. e.

[*Always state clearly, on the same sheet of paper as the graph, the units employed.*]

Plot the graphs of the following, using an x unit twice as large as the y unit.

1. $3x + 4y = 12$. **2.** $3x - 4y = 12$. **3.** $y = 2x$.

4. $y + 3x = 0$. **5.** $5x - 2y = 1$. **6.** $2x + 2y + 2 = 0$.

Plot graphs of the following using a y unit ten times as large as the x unit.

7. $x + y = 11$. **8.** $x - 2y = 20$. **9.** $10x = y$. **10.** $20x + y = 0$.

Trace graphs of the equation $y = x^2$.

11. When the x unit is five times as large as the y unit.

12. .... four ....

Trace graphs of the equation $x^2 = y$.

13. When the x unit is equal to the y unit.

14. .... ten times as large as the y unit.

15. .... five ....

Trace graphs of the equation $y = 4x^2$.

16. When the x unit is equal to the y unit.

17. .... four times the y unit.

18. Construct a graph to show the square roots of numbers from 49 to 64. From it write down (correct to two decimal places) the square roots of 53·6, 57·8, 59·5, 61·6.

Verify one of your results by the Arithmetical method.

19. Construct a graph to show the square roots of numbers from 36 to 49. From it write down (correct to two decimal places) the square roots of 38·6, 39·7, 40, 42·6, 46·8.

[With the curve $y=x^2$, use 5 inches for the x unit, half an inch for the y unit.]

From the above graph read off approximate values of the squares of 6·44, 6·68, 6·82.

20. Plot the points $(7, 7^2)$, $(7{\cdot}1, 7{\cdot}1^2)$, $(7{\cdot}2, 7{\cdot}2^2)$, $(7{\cdot}3, 7{\cdot}3^2)$, $(7{\cdot}4, 7{\cdot}4^2)$. Join them and read off the square roots of 49·8, 50·7, 51·3, 53·9 correct to two decimal places.

[Use 10 inches for the x unit, one inch for the y unit.]

From the above graph write down approximate values of the squares of 7·05, 7·16, 7·28, 7·36.

21. Find from one graph, correct to two decimal places, the square roots of 54·6, 58·8, 62·4.

Verify one root by the Arithmetical method.

22. Plot the points $(8, 8^2)$, $(8{\cdot}1, 8{\cdot}1^2)$, $(8{\cdot}2, 8{\cdot}2^2)$. Join them and use the graph to determine, to one decimal place, the square roots of 6430, 6680.

23. Using 5 inches (or 10 centimetres) to denote ·1 in the x axis, and ·5 inches (or 10 centimetres) to denote unity in the y axis, plot the points $(8, 64)$, $(8{\cdot}1, 8{\cdot}1^2)$. Join them by a straight line. Assuming this straight line to be part of the graph of $y=x^2$, use it to determine the square roots (to two decimal places) of 6425, 6437, 6486.

Verify one of your results by the Arithmetical method.

In each of the following examples, use a single graph to determine the square roots of the given numbers (use large units).

In each case verify one answer by the Arithmetical method.

24. 81·96, 82·6, correct to three decimal places.

25. 8346, 8424, „ „ two „ „

26. 101·68, 100·96, „ „ three „ „

27. 152·8, 167·6, „ „ two „ „

Use one of the methods of Art. 136 to find the approximate values of the following:

28. $\sqrt{3}$. **29.** $\sqrt{6}$. **30.** $\sqrt{7}$. **31.** $\sqrt{11}$. **32.** $\sqrt{5{\cdot}6}$.

33. $\sqrt{1{\cdot}8}$. **34.** $\sqrt{6{\cdot}6}$. **35.** $\sqrt{4{\cdot}5}$. **36.** $\sqrt{5{\cdot}7}$. **37.** $\sqrt{4{\cdot}3}$.

38. Draw a graph to find the cube root of any number between 125 and 216. Write down the cube roots of 144 and 198 correct to two decimal places.

39. Draw enough of the graph of $y=x^3$ to find the cube roots of numbers between 8 and 27.

Write down the cube roots of 15 and 21 correct to two decimal places.

40. Find the cube root of 8·25 correct to two decimal places. Test your result.

[Plot the points $(2, 2^3)$ $(2{\cdot}1, 2{\cdot}1^3)$, using a large x unit, say 5 inches, to denote ·1. Join the points by *a straight line*, and assume this straight line to be part of the graph of $y=x^3$.]

Find the cube roots of the following, correct to two decimal places:

41. 27·9. **42.** 28·6. **43.** 29·2 **44.** 30. **45.** 65·6.

46. 67·8. **47.** 68·5. **48.** 127. **49.** 128·8. **50.** 130

CHAPTER XXV.

QUADRATIC EQUATIONS.

139. When an equation contains the square of the unknown quantity, and no higher power, it is called a **quadratic equation,** or an **equation of the second degree.**

$$\left.\begin{aligned} x^2 - 7x + 12 &= 0, \\ 6x^2 &= 7x + 3, \\ 12 &= 23x - 5x^2, \\ x^2 - 4 &= 0 \end{aligned}\right\} \text{ are examples of such.}$$

140. Solution of quadratics by factorization.

Let us consider the equation $x^2 - 7x + 12 = 0$.

It may be written $(x-3)(x-4) = 0$.

We notice that when $x = 3$,

$$\begin{aligned} \text{the left hand side} &= (3-3)(3-4) \\ &= 0 \times (-1) = 0, \end{aligned}$$

i.e. the equation is satisfied, or 3 is a root of the equation.

Also when $x = 4$,

$$\begin{aligned} \text{the left hand side} &= (4-3)(4-4) \\ &= 1 \times 0 = 0, \end{aligned}$$

$\therefore$ 4 also is a root of the equation.

It will be proved later on that every quadratic equation has two roots and only two.

N.B.—Every *multiple* of 0 is 0.

$$6 \times 0 = 0, \qquad 1000 \times 0 = 0,$$
$$0 \times a = 0, \qquad 0 \times x^3 = 0.$$

Examples. XXV. a.

Write down the roots of the following equations:

1. $(x-1)(x-2)=0$
2. $(x-1)(x+1)=0$.
3. $(x-a)(x-b)=0$
4. $x(x-1)=0$
5. $(x+2)(x+3)=0$.
6. $(x+a)(x-b)=0$
7. $(x+2)x=0$
8. $(x-2a)(x-b)=0$.
9. $(x+a)(x-2b)=0$.
10. $(x-\frac{1}{2})(x+\frac{3}{4})=0$.
11. $(x+\frac{1}{5})(x+\frac{3}{8})=0$.
12. $x(x+\frac{1}{3})=0$.
13. $\left(x-\frac{a}{2}\right)\left(x-\frac{b}{3}\right)=0$
14. $(x-\overline{a+b})(x-\overline{a-b})=0$.

Write down the roots of the following equations:

15. $\left(x-\frac{a+b}{2}\right)\left(x+\frac{c+d}{2}\right)=0.$

16. $(x-\overline{p-2q})(x-\overline{2p-q})=0.$

17. $\{x-2(a+b)\}\{x+3(a-b)\}=0.$

18. $(x-a^2)(x+b^2)=0.$

19. $\{x+(a-b)^2\}\{x-(a+b)^2\}=0.$

20. $(x-3)^2=0.$

21. $x(x-a)=0.$

22. $x(x+4)=0.$

23. $(x+a)^2=0.$

24. $(x+2a)^2=0.$

141. *Solve the equation* $x^2=x+20.$

Transposing all the terms to the left-hand side (or subtracting $x+20$ from both sides)

$$x^2-x-20=0,$$

factorizing,

$$(x-5)(x+4)=0;$$

$$\therefore\ x=5 \text{ or } -4.$$

Verification. When $x=5$, $x^2-x-20=25-5-20$
$=0;$

$\therefore$ 5 is a root of the equation.

When $x=-4$, $x^2-x-20=(-4)^2-(-4)-20$
$=16+4-20=0:$

$\therefore$ -4 is also a root.

Solve the equation $4x^2-16x=84.$

Transposing 84 to the left hand side,

$$4x^2-16x-84=0.$$

Dividing both sides by 4, $x^2-4x-21=0,$

factorizing,

$$(x-7)(x+3)=0;$$

$$\therefore\ x=7 \text{ or } -3.$$

Verification. When $x=7$

$$\begin{aligned}4x^2-16x-84&=4\times 49-16\times 7-84\\&=196-112-84\\&=0;\end{aligned}$$

$\therefore$ 7 is a root of the equation.

When $x=-3$,

$$\begin{aligned}4x^2-16x-84&=4\times 9-16(-3)-84\\&=36+48-84\\&=0;\end{aligned}$$

$\therefore$ -3 is also a root.

142. When an equation contains the square of the unknown quantity, and no first power of the unknown quantity, it is called

a **pure quadratic.** If it contains both the square and the first power of the unknown, it is called an **adfected quadratic.**

$x^2-4=0$ and $6x^2=54$ are examples of pure quadratics.

$x^2-7x+12=0$ is an adfected quadratic.

Pure quadratics are easily solved by factorization.

Solve the quadratic $\qquad 6x^2=54.$

Dividing both sides by 6, $\qquad x^2=9.$

Adding 9 to both sides, $\qquad x^2-9=0,$

i.e. $(x-3)(x+3)=0,$

$\therefore\ x=3$ or $-3.$

Or we might proceed thus,

$x^2=9$ as before.

Taking the square root of each side

$x=\pm 3.$

143. *Solve the equation* $x^2=12-x.$

Transposing all terms to the left-hand side (or subtracting $12-x$ from both sides),

the equation becomes $\qquad x^2+x-12=0.$

Factorizing, $\qquad (x+4)(x-3)=0,$

from which we see that -4 and 3 are the roots reqd.

Verification. When $\qquad x=-4,$

the left-hand side $=(-4)^2=16,$

the right-hand side $=12-(-4)=16$;

$\therefore\ -4$ is a root.

When $x=3$, the left-hand side $=(3)^2=9,$

the right-hand side $=12-3=9$;

$\therefore\ 3$ is also a root.

Examples. XXV. b.

Solve the following equations, verifying the solutions in each case:

1. $x^2-7x+10=0.$	**2.** $x^2-5x+6=0.$	**3.** $x^2-4=0.$
4. $x^2-3x=0.$	**5.** $x^2+4x+3=0.$	**6.** $x^2+4x-5=0.$
7. $x^2=8x-7.$	**8.** $x^2-2=x.$	**9.** $x^2-3=1.$
10. $x^2+10=11x.$	**11.** $4x=45-x^2.$	**12.** $12x-27=x^2.$
13. $x^2=20-x.$	**14.** $x^2=7x.$	**15.** $2x^2-1=1.$
16. $x^2-4x+4=0.$	**17.** $x^2+3x=0.$	**18.** $21+10x+x^2=0.$

Solve the following equations, verifying the solutions in each case:

19. $14x+15=x^2$.
20. $40=3x+x^2$.
21. $x^2+225=30x$.
22. $2x^2-3=15$.
23. $4x^2=8x$.
24. $3x^2+21x=0$.
25. $103x=x^2+102$.
26. $x^2+16x+15=0$.

144. Let us take the equation $2x^2-11x+12=0$.

It may be written $(2x-3)(x-4)=0$.

We see that if $2x-3=0$, *i.e.* if $x=\frac{3}{2}$, the equation is satisfied, for $0\times(\frac{3}{2}-4)=0$.

Also if $x-4=0$, *i.e.* if $x=4$, the equation is again satisfied;

$\therefore$ $\frac{3}{2}$ and 4 are the roots of the equation.

Solve the equation $\quad x^2=2(x+12)$.

Removing the brackets $x^2=2x+24$.

Transposing all terms to the left-hand side,

$$x^2-2x-24=0.$$

Factorizing, $\quad (x-6)(x+4)=0$;

$\therefore$ 6 and -4 are the reqd. roots.

Solve the equation $\quad x^2-4x+4=0$.

Factorizing, $\quad (x-2)(x-2)=0$;

$\therefore$ in this case the roots are equal and each of them is 2.

145. If fractions or brackets occur in the given equation, they should first be cleared away.

Example 1. Solve the equation $3x-8=\dfrac{x^2}{4}$.

Multiplying both sides by 4, $\quad 12x-32=x^2$.

Transposing all terms to the left hand side (or subtracting x^2 from both sides),

$$12x-32-x^2=0.$$

Re-arranging and changing signs throughout [this is permissible, for if $a=b$, $-a=-b$; if $a=0$, $-a=0$],

$$x^2-12x+32=0.$$

Factorizing, $\quad (x-4)(x-8)=0$;

$\therefore$ 4 and 8 are the reqd. roots, or $x=4$ or 8.

Verification. When $x=4$, the left-hand side $=3\times4-8=4$.

.... the right-hand side $=\dfrac{(4)^2}{4}=4$;

$\therefore$ 4 is a root.

When $x=8$, the left-hand side $=3\times8-8=16$.

.... the right-hand side $=\dfrac{(8)^2}{4}=\dfrac{64}{4}=16$;

$\therefore$ 8 is also a root.

Example 2. Solve the equation $\dfrac{7}{3x-1}-\dfrac{4}{x+1}=\dfrac{1}{4}$.

Multiplying both sides by $4(3x-1)(x+1)$, the L.C.M. of the denominators,

$$28(x+1)-16(3x-1)=(x+1)(3x-1),$$

$$28x+28-48x+16=3x^2+2x-1.$$

Transposing and arranging, $-3x^2-22x+45=0$,

$$3x^2+22x-45=0,$$

$$(3x-5)(x+9)=0;$$

$\therefore$ $\frac{5}{3}$ and -9 are the reqd. roots.

It is important to observe **that if $x-a$ is a factor of both sides of an equation, a is a root of the equation.**

This is at once seen by substitution.

Example 3. Solve the equation $2(2x-5)+7x(2x-5)=0$.

$2x-5$ is a factor throughout; $\therefore$ $2x-5=0$ gives a root whence $x=\frac{5}{2}$.

Having divided by $2x-5$, we have left

$$2+7x=0$$

whence $x=-\frac{2}{7}$;

$\therefore$ the reqd. roots are $\frac{5}{2}$ and $-\frac{2}{7}$.

Examples. XXV. c.

Write down the roots of the following quadratic equations:

1. $(2x-3)(x-4)=0$.
2. $(3x+1)(2x-1)=0$.
3. $(3x+4)(5x+6)=0$.
4. $x(7x+9)=0$.
5. $(5x-7)(6x+1)=0$.
6. $(7x-8)^2=0$.
7. $(2x-a)(2x-b)=0$.
8. $(5x+a)(6x+b)=0$.
9. $(2x-\overline{a+b})(3x-\overline{c+d})=0$.
10. $3(4x+5)(2x-9)=0$.

Solve the following equations:

11. $x^2=2-x$.
12. $8x-x^2=15$.
13. $x^2=4(x+8)$.
14. $2(5x-12)=x^2$.
15. $x(x-4)=5$.
16. $4x^2=1$.
17. $x^2-4x=4(x-4)$.
18. $1+2x^2=3x$.
19. $x(x+4)=6(x+4)$.
20. $5x^2+17x=0$.
21. $x-10=x(x-10)$.
22. $4x(x+1)+1=0$.
23. $x^2+4{\cdot}8x+2{\cdot}87=0$.
24. $x+\dfrac{1}{x}=2$.
25. $x-\dfrac{9}{2}+\dfrac{2}{x}=0$.
26. $(2x-1)(3x+1)=11$.
27. $2x^2+\dfrac{13x}{2}=6$.
28. $5x(2x-3)+7(2x-3)=0$.
29. $x-1=\dfrac{2}{x}$.
30. $(2x+1)(x+8)=27$.
31. $\dfrac{x+10}{x-5}-\dfrac{10}{x}=\dfrac{11}{6}$.
32. $150x^2=299x+2$.
33. $(5x-3)(3x+1)=1$.
34. $6(4x+5)+\dfrac{7}{x}(4x+5)=0$.
35. $13x^2-6x-7=0$.
36. $x+35=70x^2$.
37. $9x^2=18x+16$.
38. $\dfrac{1}{x-1}-\dfrac{1}{x+3}=\dfrac{1}{35}$.

SOLUTION OF QUADRATICS BY COMPLETING SQUARES.

146. Take the equation $a^2+2ab=0$.

Adding b^2 to both sides, $a^2+2ab+b^2=b^2$,

$$i.e.\quad (a+b)^2=b^2.$$

The addition of b^2 to both sides **completed the square** on the left-hand side.

Take the equation $x^2-6x=0$.

Adding 9 to both sides, $x^2-6x+9=9$,

$$(x-3)^2=3^2.$$

Again the left-hand side becomes a complete square.

More generally, to complete the square on the left of the equation $x^2-2ax=0$ we must add a^2 to both sides.

The equation becomes $x^2-2ax+a^2=a^2$,

$$\text{or } (x-a)^2=a^2.$$

x^2+8x becomes $(x+4)^2$ by adding 16, *i.e.* 4^2.............(1)

x^2-2cx .... $(x-c)^2$ .... c^2. ....(2)

x^2+10x .... $(x+5)^2$ .... 5^2. ....(3)

Thus we observe that any expression of the form $x^2\pm 2px$ becomes a complete square when **we add the square of half the coefficient of x**.

In (1) we add $\left(\frac{8}{2}\right)^2$.

In (2) .... $\left(-\frac{2c}{2}\right)^2$.

In (3) .... $\left(\frac{10}{2}\right)^2$.

147. Let us now employ this to solve quadratic equations.

Example 1. Solve the quadratic $x^2+4x=32$.

Adding the sq. of half the coeff. of x to both sides,

$$x^2+4x+\left(\frac{4}{2}\right)^2=32+\left(\frac{4}{2}\right)^2,$$

$$i.e.\quad x^2+4x+(2)^2=36,$$

$$(x+2)^2=36.$$

Taking the square root of both sides,

$$\therefore\quad x+2=\pm 6 \text{ .... (i)}$$

With the positive sign $x+2=6$,

$$x=4.$$

With the negative sign $\qquad x+2=-6,$

$$x=-8;$$

$\therefore$ 4 and -8 are the reqd. roots.

In connection with (i) we at first sight think we ought to say

$$\pm(x+2)=\pm 6,$$

for $\pm(x+2)$ is the sq. root of $(x+2)^2$ just as ± 6 is the sq. root of 36.

This however is unnecessary, as we see if we take the *four* different cases separately.

$$\left.\begin{array}{ll}\text{With positive signs on both sides, } & x+2=6, \quad x=4 \\ \text{..... negative ....} & -x-2=-6, \; x=4\end{array}\right\} \text{the same result.}$$

With the positive sign on the left and the negative sign on the right,

$$x+2=-6, \; x=-8.$$

With the negative sign on the left and the positive sign on the right,

$$-x-2=+6,$$

$$x+2=-6, \; x=-8, \text{ again the same result.}$$

Thus it is sufficient if we attach the double sign ($\pm$) to one side.

We always attach it to the numerical square root.

148. Before completing squares the coefficient of x^2 must be reduced to unity.

Solve the equation $\qquad 22-x=6x^2.$

Re arranging by transposition, $6x^2+x=22.$

Dividing both sides by 6 to make the coefficient of x equal to unity,

$$x^2+\frac{x}{6}=\frac{22}{6}.$$

Adding the sq. of half the coeff. of x; *i.e.* $\left(\frac{1}{12}\right)^2$, to both sides,

$$x^2+\frac{x}{6}+\left(\frac{1}{12}\right)^2=\frac{22}{6}+\frac{1}{144},$$

$$\left(x+\frac{1}{12}\right)^2=\frac{528+1}{144}$$

$$=\frac{529}{144},$$

Taking the sq. root of both sides,

$$x + \frac{1}{12} = \pm \frac{23}{12}.$$

With the positive sign $\quad x + \frac{1}{12} = \frac{23}{12},$

$$x = \frac{23 - 1}{12} = \frac{11}{6}.$$

With the negative sign $\quad x + \frac{1}{12} = -\frac{23}{12},$

$$x = \frac{-23 - 1}{12} = -2;$$

$$\therefore \ \frac{11}{6} \text{ and } -2 \text{ are the reqd. roots}$$

149. *To solve the general quadratic $ax^2 + bx + c = 0$.*

$$ax^2 + bx = -c,$$

$$x^2 + \frac{bx}{a} = -\frac{c}{a}.$$

Adding the square of half the coeff of x to both sides,

$$^2 + \frac{bx}{a} + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$$

$$= \frac{b^2 - 4ac}{4a^2}.$$

Taking the sq. root of both sides,

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a},$$

$$x = - \qquad 2a \qquad .$$

The above formula may be used for the solution of any quadratic equation.

There are therefore three methods of solving quadratics:

(1) by factorization, (2) by completing squares,

(3) by using the formula $\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

The student should have considerable practice in all three methods.

When the factors cannot be seen *readily*, the second or third method should be employed.

Examples. XXV. d.

Solve the equations:

1. $6x^2=2-x$.

2. $1-26x^2=11x$.

3. $x+1=156x^2$.

4. $5x^2=4x+1$.

5. $3x^2+10=17x$.

6. $7x^2+32x=15$.

7. $2x^2+19x+9=0$.

8. $(x-1)^2=16$.

9. $2(x^2+1)-5x=0$.

10. $11x=3(2x^2+1)$.

11. $3(x-1)(x+1)=8x$.

12. $(x-1)(x+1)=\frac{7x}{12}$.

13. $15=4(3x^2+2x)$.

14. $(2x-1)^2=25$.

15. $(3x-\frac{1}{2})^2=49$.

16. $3x(5x-1)=4(x+9)$.

17. $25x^2-7x=86$.

18. $5x-11=x(5x-11)$.

19. $13x+9=10x^2$.

20. $\left(\frac{x}{2}-5\right)^2-36=0$.

21. $3(3x+4)+5x(3x+4)=0$.

22. $\frac{2x-3}{2}=\frac{4x-6}{x}$.

23. $x(x-1)+\frac{1}{2}(x-1)=0$.

24. $\frac{2x-3}{5}+\frac{2(2x-3)}{3x}=0$.

25. $7(3x-6)+11x(2x-4)-3x(5x-10)=0$.

26. $\frac{2}{3(x-1)}-\frac{3}{2x+1}=\frac{1}{15}$.

27. $\frac{6}{x-2}=\frac{5}{x-4}-\frac{6}{x-3}$.

28. $\frac{x-1}{x+1}+\frac{x-3}{x+3}=\frac{2x+1}{2x+2}$.

29. $\frac{x}{5+x}+\frac{7}{6-4x}=\frac{x-7}{x-6}$.

30. $\frac{3x+4}{5}-\frac{30-2x}{x-6}=\frac{7x-14}{10}$.

31. $\frac{2}{x-2}-\frac{3}{x-3}=\frac{4}{x-4}-\frac{5}{x-5}$.

32. $\frac{2}{x+3}+\frac{x+3}{2}=\frac{10}{3}$.

33. $\frac{2x}{x-1}+\frac{3x-1}{x+2}-\frac{5x-11}{x-2}=0$.

34. $\frac{x-3}{x+3}-\frac{x+3}{x-3}+6\frac{6}{7}=0$.

When the quantity under the radical sign ($\sqrt{\ }$) is not a perfect square, the *approximate* values of the roots should be found by finding the square root to a few decimal places.

Thus if $$x=\frac{9\pm\sqrt{21}}{10},$$

$$x=\frac{9\pm 4\cdot583\ldots}{10}\quad(\text{for }\sqrt{21}=4\cdot583\ldots)$$

$$=1\cdot36,\text{ or }\cdot44,\text{ correct to two decimal places.}$$

Examples. XXV. e.

When the exact values of the roots of the following equations cannot be found, give results *correct to two decimal places*, *i.e.* to the nearest hundredth.

Solve

1. $x^2-2x=1$.
2. $x^2=2(1-x)$.
3. $x(x-3)=x-1$.
4. $x=\frac{x+4}{x-1}$.
5. $5x^2-9x-4=0$.
6. $\frac{x+1}{x+2}+\frac{x-3}{x-4}=0$.
7. $x^2=\sqrt{3}(2x-\sqrt{3})$.
8. $\frac{1}{x+3}+\frac{1}{x+6}+\frac{1}{x+9}=0$.
9. $\frac{2x-1}{3x+2}+\frac{x-3}{x+1}=0$
10. $\frac{x-1}{x^2+3x+2}+\frac{x-3}{x^2+5x+6}=\frac{1}{x+2}$.
11. $2(x-1)=\frac{4-5x}{x+1}$.
12. $\frac{1}{x-2}+\frac{1}{x-3}+\frac{1}{x-4}=0$.
13. $\frac{3x+1}{3x-1}-\frac{3x-1}{3x+1}=2$.
14. $x^2-\sqrt{3}x-6=0$.

MISCELLANEOUS FORMS OF QUADRATIC EQUATIONS.

***150. Example 1.** Solve $\frac{x+2}{x-2}-\frac{x-3}{x+3}=\frac{x+4}{x-4}-\frac{x-1}{x+1}$.

Simplifying each side separately,

$$\frac{x^2+5x+6-(x^2-5x+6)}{(x-2)(x+3)}=\frac{x^2+5x+4-(x^2-5x+4)}{(x-4)(x+1)},$$

$$\frac{10x}{x^2+x-6}=\frac{10x}{x^2-3x-4};$$

$$\therefore\ x=0\ \text{ or }\ \frac{1}{x^2+x-6}=\frac{1}{x^2-3x-4},$$

$$i.e.\quad x^2+x-6=x^2-3x-4.$$
$$4x=2,$$
$$x=\tfrac{1}{2};$$
$\therefore$ 0, $\frac{1}{2}$ are the reqd. solutions.

*Examples. XXV. f.

Solve the equations:

1. $x^4+100=29x^2$.

[Treat the equation as a quadratic for x^2.]

2. $x^2+\dfrac{324}{x^2}=45$.

3. $x^3+\dfrac{27}{x^3}=28$.

4. $\dfrac{x+2}{x-2}-\dfrac{x}{x+5}=\dfrac{x+3}{x-3}-\dfrac{x}{x+4}$.

5. $x^2-2x+\dfrac{36}{x^2-2x}=15$.

[Let $x^2-2x=v$, and first solve for v. Two values of v will be found, and we shall therefore have *four* values of x.]

6. $x^2-1+x^3-x=0$.

[Factorize the left-hand side.]

7. $5x^3-4x^2=5x-4$.

8. $x^2-4x-4=\dfrac{5}{x^2}-\dfrac{4}{x}$.

9. $x-\dfrac{1}{x}=\dfrac{\;}{21}\left(x^3-\dfrac{1}{x^3}\right)$.

10. $(x+1)(x+2)(x+3)(x+4)=24+34(x^2+5x)$.

11. $6x^3+(5-x)^3=5(5+x)(5+2x)$.

12. $(x+1)(x+2)(x+3)(x+4)=24$.

13. $\dfrac{x-1}{x+1}+\dfrac{x-4}{x+4}=\dfrac{x-2}{x+2}+\dfrac{x-3}{x+3}$.

14. $x(x+1)(x+2)(x+3)=120$.

15. $x^2+3x-\dfrac{9}{2}+\dfrac{2}{x^2+3x}=0$.

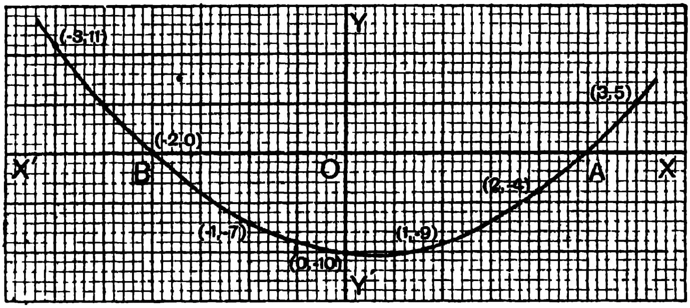
16. $16x(x+1)(x+2)(x+3)=9$.

17. $x^4+2x^3-11x^2+4x+4=0$.

CHAPTER XXVI.

GRAPHS OF QUADRATIC FUNCTIONS OF x AND GRAPHIC SOLUTIONS OF QUADRATIC EQUATIONS.

151. *Solve the equation* $2x^2 - x - 10 = 0$ *graphically.*



First Method. Let us trace the graph of $y = 2x^2 - x - 10$, using a unit for the x values 10 times as large as that for the y values, as in Art. 134.

When

$x = 0$	1	2	3	...	-1	-2	-3
$2x^2 = 0$	2	8	18	..	2	8	18
$-x - 10 = -10$	-11	-12	13	..	-9	-8	-7
$y = 2x^2 - x - 10 = -10$	-9	-4	5	...	-7	0	11

$\therefore$ (0, -10), (1, -9), (2, -4), (3, 5), (-1, -7), (-2, 0), (-3, 11) are points on the graph.

Marking these points as shown in the diagram, and drawing the curve carefully, we have the graph of $y = 2x^2 - x - 10$.

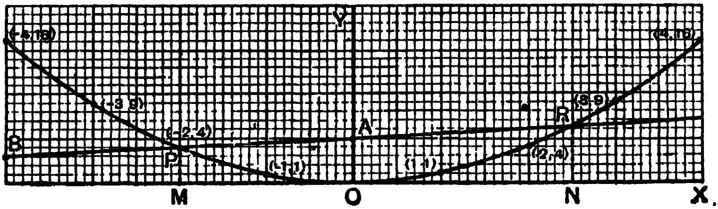
At the points A and B where this curve meets XOX′ the axis of x, $y=0$; $\therefore$ at those points $2x^2-x-10=0$.

But OA and OB are the values of x at these points;

$\therefore$ they are the roots of the given equation.

From the diagram we see that the roots are 2·5 and -2.

Second Method. First trace the graph of $y=x^2$, using a unit for the x values 10 times as large as that for the y values, as in Art. 134.



We thus obtain the curve POR as in the diagram.

Then trace in the same diagram, and *with the same units*, the graph of
$$2y-x-10=0.$$

We know this to be a straight line. (Art. 71.)

When $x=0$, $y=5$; $\therefore$ (0, 5) is a point on the straight line.

Mark this point A.

When $x=-4$, $y=3$; $\therefore$ $(-4, 3)$ is also on the line.

Mark this point B, and join AB.

The straight line AB is the graph of $2y-x-10=0$.

Mark the points P and R where this line meets the curve PQR.

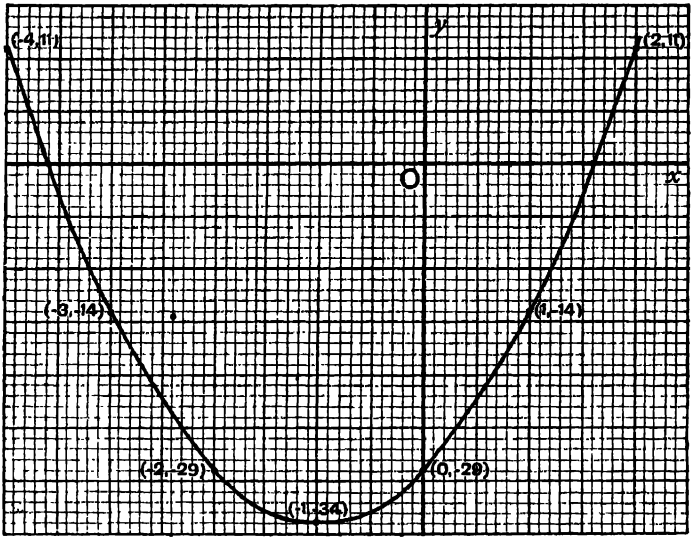
Now at the point P, the ordinate PM is the same for both graphs, *i.e.* y is the same in both the equations $y=x^2$ and $2y-x-10=0$; $\therefore$ at the point P, $2x^2-x-10=0$. OM is therefore a root of this equation. From the diagram OM $=-2$.

In precisely the same way, the ordinate at R is the same in both equations, $y=x^2$ and $2y-x-10=0$, $\therefore$ ON is another root of the equation $2x^2-x-10=0$. From the diagram ON $=2·5$;

$\therefore$ the reqd. roots are -2 and 2·5.

152. *Find graphically, correct to one decimal place, the roots of the equation* $5x^2+10x-29=0$.

Trace the graph of $y=5x^2+10x-29$.



When

$x=0$	1	2	3
$5x^2=0$	5	20	45
$10x-29=-29$	-19	-9	1
$y=-29$	-14	11	46

When

$x=-1$	-2	-3	-4
$5x^2=5$	20	45	80
$10x-29=-39$	-49	-59	-69
$y=-34$	-29	-14	11

Plotting the points $(0, -29)$ $(1, -14)$ $(2, 11)$ $(-1, -34)$ $(-2, -29)$ $(-3, -14)$ $(-4, 11)$ and taking the x unit ten times as large as the y unit, we have the curve as shown in the diagram.

The equation is satisfied when $5x^2 + 10x - 29 = 0$, *i.e.* when $y = 0$, *i.e.* where the curve cuts the axis of x.

From the diagram, the roots required are

$$1 \cdot 6, \quad -3 \cdot 6.$$

Verification. When $x = 1 \cdot 6$,
$$\begin{aligned} 5x^2 + 10x - 29 &= 5(2 \cdot 56) + 16 - 29 \\ &= 12 \cdot 8 + 16 - 29 \\ &= -\cdot 2. \end{aligned}$$

Thus when $x = 1 \cdot 6$, $5x^2 + 10x - 29$ is nearly zero.

$\therefore$ $1 \cdot 6$ is an approximate root. In the same way we can verify the fact that $-3 \cdot 6$ is an approximate root.

If we trace the graphs of $y = x^2$ and $y = x^2 + bx + c$, where b and c have any assigned values, using the same units in each case, we shall obtain the same curve in different positions. This is easily seen by cutting out one curve and superimposing it on the other.

In general, it will be found that the graph of any equation in two variables, whose terms of the second degree form a perfect square, is a parabola.

For instance, if we plotted a number of points on the curve $(2x + 3y)^2 + 3x - 2y + 5 = 0$ and joined them by an even curve we should obtain a parabola.

MAXIMUM AND MINIMUM VALUES OF QUADRATIC EXPRESSIONS OF ONE VARIABLE.

153. These all hinge upon the fact that a perfect square is always positive, *i.e.* it cannot be less than zero.

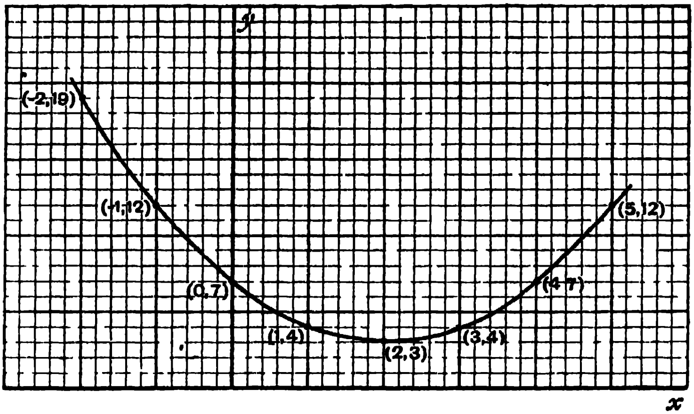
To find the minimum value of $x^2 - 4x + 7$ for real values of x.

$$x^2 - 4x + 7 = (x - 2)^2 + 3.$$

$\therefore$ the given expression is least when $(x - 2)^2 = 0$.

The reqd. minimum value is therefore 3.

To find the minimum value of $x^2 - 4x + 7$ graphically.



Let us trace the graph of $y = x^2 - 4x + 7$.

When

$x = -2$	-1	0	1	2	3	4	5
$x^2 + 7 = 11$	8	7	8	11	16	23	32
$-4x = 8$	4	0	-4	-8	-12	-16	-20
$y = 19$	12	7	4	3	4	7	12

Plotting the pts. $(-2, 19)$ $(-1, 12)$ $(0, 7)$ $(1, 4)$ $(2, 3)$ $(3, 4)$ $(4, 7)$ $(5, 12)$ and joining them by an even curve, we have the curve shown in the diagram

From it we see that the minimum value of y, *i e.* of $x^2 - 4x + 7$, is 3

[In the diagram the x unit is taken five times as large as the y unit]

To find the maximum value of $3{\cdot}5 + 4x - 4x^2$ *for real values of* x.

$$3{\cdot}5 + 4x - 4x^2 = 4{\cdot}5 - (1 - 4x + 4x^2)$$
$$= 4{\cdot}5 - (1 - 2x)^2.$$

$\therefore$ the given expression is greatest when $(1-2x)^2$ is least, *i.e.* when $1-2x=0$.

Hence 4·5 is the maximum value reqd.

By plotting the graph of $y=3\cdot5+4x-4x^2$, we can find the maximum value graphically, as in the preceding example.

154. *Between what values of x is the expression* $19x-2x^2-35$ *positive?*

Let y denote the given expression.

$$y=-(2x^2-19x+35)=-(2x-5)(x-7)$$
$$=(2x-5)(7-x)=2\left(x-\tfrac{5}{2}\right)(7-x).$$

When $x<2\frac{1}{2}$, $x-\frac{5}{2}$ is negative and $7-x$ is positive;

$\therefore$ y is negative.

When $x>2\frac{1}{2}$ but <7, $x-\frac{5}{2}$ is positive and $7-x$ is positive;

$\therefore$ y is positive.

When $x>7$, $x-\frac{5}{2}$ is positive and $7-x$ is negative;

$\therefore$ y is negative.

$\therefore$ the given expression is only positive as long as x is between $2\frac{1}{2}$ and 7.

This may be seen graphically by plotting the curve

$$y=19x-2x^2-35.$$

Examples XXVI.

1. Draw the graph of $3x^2-5x-3$ for the following values of x, -2, -1, 0, 1, 2, 3,

(i) Using an x unit ten times as large as the y unit.
(ii) .... five ... .

2. Draw the graph of $5x^2+4x-21$.

(i) Using an x unit ten times as large as the y unit.
(ii) .... five ... .

3. Draw the graph of x^2-4x.

(i) Using an x unit ten times as large as the y unit.
(ii) ... . five ....

4. Draw the graph of $4(x^2-1)$.

(i) Using an x unit ten times as large as the y unit.
(ii) .... five .. . .

[*Tabulate values of x and y before choosing your units.*]

5. Prove graphically that the expression $x^2-6x+13$ is positive for all real values of x.

6. Show graphically that the expression $4x-6-x^2$ is never positive for real values of x.

Solve the following equations graphically:

7. $4x^2-4x-15=0$.
8. $4x^2-4x-35=0$.
9. $x^2+1\cdot1x-\cdot8=0$.
10. $x^2-3\cdot3x+2=0$.
11. $6x^2-23x+21=0$, to the nearest tenth.
12. $10x^2+21x-13=0$.
13. $5x^2-3x-16=0$, to the nearest tenth.

14. Draw the graph of $4x^2-4x+1$. What do you deduce as to the roots of the equation $4x^2-4x+1=0$?

15. Plot the graph of $4x^2-3x+7$ using integral values of x from -2 to 3. What do you deduce as to the roots of the equation $4x^2-3x+7=0$?

16. Prove graphically that the expression $13-6x-x^2$ is never greater than 22 for real values of x.

17. Draw the graph of x^2-3x, and deduce approximate values of the roots of the equation $x^2-3x=3$.

18. Plot the graph of $5x^2-3x-24$, and from it deduce the roots of the equation $5x^2=3x+26$.

19. Draw the graphs of $y=x^2$, $2y=3x+14$ in the same diagram, and deduce the roots of the equation $2x^2-3x-14=0$.

20. Draw the graphs of $y=x^2$ and $5y-8x-69=0$ and deduce the roots of the equation $5x^2=8x+69$.

21. In the equation $y=5x^2-4x-10$, find the corresponding values of y to the values -2, -1, 0, 1, 2, 3 of x. Draw the portion of the curve thus given, and deduce approximate values of the roots of the equation $5x^2-4x-10=0$. Read off the minimum value of the expression $5x^2-4x-10$.

22. Find graphically the values of x for which the expression x^2-x-6 vanishes. Prove that for all values of x between these limits the expression is negative and for all other real values of x positive.

23. Draw the graphs of $y=x^2$ and $2y-3x-20=0$, and deduce the roots of the equation $2x^2=3x+20$.

24. Draw the graph of $y=(x-2)(x-3)$, and deduce approximate roots of the quadratic $(x-2)(x-3)=5$.

25. In the equation $y=3+3x-5x^2$, find the values of y corresponding to the values $-0\cdot4$, $-0\cdot2$, 0, $0\cdot2$, $0\cdot4$, $0\cdot6$ of x. Plot the points thus obtained, using an inch to represent $0\cdot2$ along the axis of x, and an inch to represent unity along the axis of y. Write down the maximum value of y.

26. Prove graphically that the line $y=6x-13$ meets the curve $y=x^2-4$ at one point only. Find its co-ordinates, and verify your result algebraically.

27. Find graphically, as accurately as you can, the minimum value of $4x^2-3x+2$ for real values of x. Verify your result algebraically.

28. Find graphically the maximum value of $6x-3-x^2$. Verify your result algebraically.

29. Find graphically the minimum value of $x^2-5x+\frac{35}{4}$. Verify your result algebraically and write down the corresponding value of x.

30. Find graphically the minimum value of $3x^2-6x+5\cdot6$. Verify by algebra, and write down the corresponding value of x.

31. Find graphically the value of x which will give $2{\cdot}4+40x+5x^2$ a minimum value.

32. Find graphically between what limits the value of x must lie if $25x^2-30x-91$ is negative.

33. Between what limits must the value of x lie if the expression $20-2x^2-3x$ is positive? Find the limits graphically and by algebra.

CHAPTER XXVII.

SIMULTANEOUS QUADRATIC EQUATIONS.

155. In this chapter we shall consider simultaneous equations, where one at least is of a higher degree than the first.

The methods of solution are various, but the student should endeavour to reduce the equations to the forms

$$ax+by=c,$$
$$ax-by=c'.$$

Addition and subtraction will then effect the solution.

Example 1. Solve the equations $25x^2-y^2=84$, $5x-y=6$.

By division, $5x+y=14$.
Also $5x-y=6$.
Adding, $10x=20$, $\therefore\ x=2$.
Subtracting, $2y=8$, $\therefore\ y=4$.
$x=2$, $y=4$ is the reqd. solution.

Example 2. Solve the equations

$$3x+y=9, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$
$$xy=6. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

Squaring equation (1) $9x^2+6xy+y^2=81$.
From (2) $12xy=72$.
Subtracting, $9x^2-6xy+y^2=9$.
Taking the sq. root, $3x-y=\pm 3$.
We now have the two cases,

$$\left.\begin{aligned}3x+y&=9,\\3x-y&=3.\end{aligned}\right\} \qquad \begin{aligned}3x+y&=9,\\3x-y&=-3.\end{aligned}$$

Adding, $6x=12$, $x=2$. $\qquad 6x=6$, $x=1$.
Subtracting, $2y=6$, $y=3$. $\qquad 2y=12$, $y=6$.

$$\therefore \left.\begin{aligned}x&=2\\y&=3\end{aligned}\right\} \text{ and } \left.\begin{aligned}x&=1\\y&=6\end{aligned}\right\} \text{ are the reqd. solutions.}$$

Example 3. Solve the equations $9x^2+y^2=52, \quad \text{..................} \quad (1)$

$$xy=8. \quad \text{..................} \quad (2)$$

From (2) $\quad 6xy=48. \quad \text{..................} \quad (3)$

Adding to (1) *to complete the square*,

$$9x^2+6xy+y^2=100.$$

Taking the sq. root $\quad 3x+y=\pm 10.$

Also, in the same way, subtracting (3) from (1),

$$9x^2-6xy+y^2=4.$$

$$\therefore \ 3x-y=\pm 2.$$

There are now four cases,

$$\left.\begin{aligned}3x+y&=10,\\3x-y&=2.\end{aligned}\right\} \quad \left.\begin{aligned}3x+y&=10,\\3x-y&=-2.\end{aligned}\right\} \quad \left.\begin{aligned}3x+y&=-10,\\3x-y&=2.\end{aligned}\right\} \quad \begin{aligned}3x+y&=-10.\\3x-y&=-2.\end{aligned}$$

Adding, $\quad 6x=12, \quad 6x=8, \quad 6x=-8, \quad 6x=-12,$

$\quad x=2. \quad x=\frac{4}{3}. \quad x=-\frac{4}{3}. \quad x=-2.$

Subtracting, $\quad 2y=8, \quad 2y=12, \quad 2y=-12, \quad 2y=-8,$

$\quad y=4. \quad y=6. \quad y=-6. \quad y=-4.$

Hence the reqd. solutions are

$$\left.\begin{aligned}x&=2,\\y&=4.\end{aligned}\right\} \quad \left.\begin{aligned}x&=\tfrac{4}{3},\\y&=6.\end{aligned}\right\} \quad \left.\begin{aligned}x&=-\tfrac{4}{3},\\y&=-6.\end{aligned}\right\} \quad \left.\begin{aligned}x&=-2,\\y&=-4.\end{aligned}\right\}$$

Example 4. Solve the equations $4x^2+y^2=17, \quad \text{..................} \quad (1)$

$$2x+y=5. \quad \text{..................} \quad (2)$$

From (2) by squaring, $\quad 4x^2+4xy+y^2=25. \quad \text{..................} \quad (3)$

.. . . (1) by subtraction, $\quad 4xy=8.$

Subtracting this from (1) $\quad 4x^2-4xy+y^2=9.$

Taking the sq. root, $\quad 2x-y=\pm 3.$

Hence $\quad \left.\begin{aligned}2x+y&=5,\\2x-y&=3.\end{aligned}\right\}$ or $\begin{aligned}2x+y&=5,\\2x-y&=-3.\end{aligned}$

Adding, $\quad 4x=8, \quad 4x=2.$

$\quad x=2. \quad x=\frac{1}{2}.$

Subtracting, $\quad 2y=2, \quad 2y=8,$

$\quad y=1. \quad y=4.$

$$\therefore \left.\begin{aligned}x&=2,\\y&=1.\end{aligned}\right\} \quad \left.\begin{aligned}x&=\tfrac{1}{2},\\y&=4.\end{aligned}\right\} \text{ are the reqd. solutions.}$$

The Examples in XXVII. a. can all be solved by substitution. The student must be careful to do the work methodically.

Example 1. $25x^2 - y^2 = 84,$ .... (1)

$5x - y = 6.$ .... (2)

From (2), $y = 5x - 6.$

$\therefore$ by substitution in (1),

$$25x^2 - (5x-6)^2 = 84,$$

whence $60x - 36 = 84, \quad \therefore \ x = 2.$

By substitution in (2), **the simpler** of the two given equations,

$$10 - y = 6, \quad \therefore \ y = 4.$$

$\therefore \left.\begin{matrix} x=2 \\ y=4 \end{matrix}\right\}$ is the reqd. solution.

Example 2. $3x + y = 9,$ .... (1)

$xy = 6.$ .... (2)

From (1), $y = 9 - 3x.$

$\therefore$ by substitution in (2), $x(9-3x) = 6,$

i.e. $3x^2 - 9x + 6 = 0,$

i.e. $x^2 - 3x + 2 = 0,$

i.e. $(x-1)(x-2) = 0;$

$\therefore \ x = 1$ or $2.$

When $x = 1$, from (1), $y = 9 - 3x = 9 - 3 = 6.$

...... $x = 2$, ... $= 9 - 6 = 3.$

$\therefore \left.\begin{matrix} x=1 \\ y=6 \end{matrix}\right\}$ and $\left.\begin{matrix} x=2 \\ y=3 \end{matrix}\right\}$ are the reqd. solutions.

Example 3. $$9x^2+y^2=52, \qquad (1)$$
$$xy=8. \qquad (2)$$

From (2), $$y=\frac{8}{x}.$$

$\therefore$ from (1), by substitution,

$$9x^2+\frac{64}{x^2}=52,$$

$$i.e. \quad 9x^4-52x^2+64=0,$$

$$i.e. \quad (9x^2-16)(x^2-4)=0.$$

$$\therefore \quad x^2=\frac{16}{9} \text{ or } 4.$$

$$\therefore \quad x=\pm\frac{4}{3} \text{ or } \pm 2.$$

When $x=\pm\frac{4}{3}$, from (2), $y=\frac{8}{x}=\pm 8\times\frac{3}{4}=\pm 6.$

... $x=\pm 2$, ...$=\pm\frac{8}{2}=\pm 4.$

Hence $\left.\begin{array}{l}x=\frac{4}{3}\\ y=6\end{array}\right\}$ $\left.\begin{array}{l}x=-\frac{4}{3}\\ y=-6\end{array}\right\}$ $\left.\begin{array}{l}x=2\\ y=4\end{array}\right\}$ $\left.\begin{array}{l}x=-2\\ y=-4\end{array}\right\}$ are the reqd. solutions.

Examples. XXVII. a.

Solve the equations:

1. $4x^2-y^2=35,$
   $2x+y=7.$
2. $x^2-y^2=21,$
   $x+y=3.$
3. $y^2-9x^2=28,$
   $y-3x=2.$
4. $x^2-xy=35,$
   $x-y=5.$
5. $4x^2+xy=51,$
   $4x+y=17.$
6. $9x-3y=3,$
   $9x^2-y^2=5.$
7. $5x-2y=12,$
   $25x^2-4y^2=96.$
8. $4x^2-25y^2=-81,$
   $4x-10y=54.$
9. $9x^2-49y^2=29,$
   $6x-14y=2.$
10. $x+y=15,$
    $xy=54.$
11. $x-y=2,$
    $xy=15.$
12. $x-y=1,$
    $xy=132.$

13. $x+y=4,$
$xy=-117.$

14. $x+y=6,$
$xy=-91.$

15. $xy=21,$
$x-y=4.$

16. $8xy=1,$
$4(x+y)=3.$

17. $4x+y=11,$
$xy=6.$

18. $5x-y=9,$
$xy=2.$

19. $3x-2y=14,$
$xy=12.$

20. $5x+4y=28,$
$xy=8.$

21. $x^2+y^2=53,$
$xy=14.$

22. $x^2+y^2=34,$
$xy=-15.$

23. $4x^2+y^2=17,$
$xy=2.$

24. $x^2+9y^2=18,$
$xy=3.$

25. $9x^2+4y^2=136,$
$xy=10.$

26. $16x^2+25y^2=544,$
$xy=12.$

27. $\frac{1}{x}+\frac{1}{y}=\frac{3}{4},$
$xy=8.$

28. $\frac{1}{x}-\frac{1}{y}=1,$
$xy=\frac{1}{6}.$

29. $\frac{1}{x}+\frac{1}{y}=\frac{14}{45},$
$x+y=14.$

30. $\frac{1}{x}-\frac{1}{y}=-\frac{2}{35},$
$x-y=2.$

31. $\frac{2}{x}+\frac{1}{y}=1,$
$xy=-1.$

32. $\frac{3}{x}+\frac{2}{y}=12,$
$xy=\frac{1}{6}.$

33. $4x-3y=26,$
$\frac{4}{y}-\frac{3}{x}=-\frac{26}{10}.$

34. $5x+7y=17,$
$\frac{5}{y}+\frac{7}{x}=8\frac{1}{2}.$

35. $x^2+y^2=53,$
$x+y=5.$

36. $x^2+y^2=\frac{5}{16},$
$x-y=\frac{1}{4}.$

37. $4x^2+y^2=104,$
$2x+y=12.$

38. $9x^2+y^2=81,$
$3x-y=9.$

39. $x^2+xy+y^2=201,$
$x+y=16.$

40. $x^2-xy+y^2=157,$
$x-y=1.$

41. $x^2+2xy+4y^2=28,$
$x+2y=6.$

42. $9x^2+xy+4y^2=91,$
$3x-2y=13.$

***156.** In the following equations, the student's aim should be to reduce the equations to one of the forms exemplified earlier in this chapter.

Example. Solve the equations

$$x^3+y^3=91, \quad\text{.......... (1)}$$

$$x^2-xy+y^2=13. \quad\text{.......... (2)}$$

Dividing, $\quad x+y=7.$ .... (3)

Squaring, $\quad x^2+2xy+y^2=49.$

$\therefore$ from (2), $\quad xy=12.$ .... (4)

Now solve equations (3) and (4) as in Example 2, Art. 155.

*Examples. XXVII. b.

Solve the equations:

1. $x^3+y^3=9,$
$x+y=3.$

2. $x^3-y^3=37,$
$x-y=1.$

3. $8x^3+y^3=280,$
$2x+y=10.$

[Divide and then proceed as in the Example worked out.]

4. $x^3-8y^3=189,$
$x-2y=9.$

5. $27x^3+8y^3=35,$
$3x+2y=5.$

6. $8x^3-27y^3=485,$
$2x-3y=5.$

7. $x^4+x^2y^2+y^4=21, \quad \ldots\ldots(1)$
$x^2+xy+y^2=3. \quad \ldots\ldots(2)$

[Dividing (1) by (2), $x^2-xy+y^2=7. \quad \ldots\ldots(3)$

Now add and subtract equations (2) and (3), and proceed as in Example 3, Art. 155.]

8. $x^4+x^2y^2+y^4=1281,$
$x^2-xy+y^2=21.$

9. $x^4+x^2y^2+y^4=481,$
$x^2-xy+y^2=13.$

10. $x^4+x^2y^2+y^4=2613,$
$x^2+xy+y^2=67.$

11. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=13,$
$\dfrac{1}{x}+\dfrac{1}{y}=5.$

12. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=41,$
$\dfrac{1}{x}-\dfrac{1}{y}=-1.$

[See Note in Example 2, Art. 60.]

13. $\dfrac{4}{x^2}+\dfrac{1}{y^2}=100,$
$\dfrac{2}{x}+\dfrac{1}{y}=13.$

14. $\dfrac{9}{x^2}+\dfrac{1}{y^2}=\dfrac{26}{25},$
$\dfrac{3}{x}-\dfrac{1}{y}=\dfrac{4}{5}.$

15. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=61,$
$30xy=1$

16. $\dfrac{1}{x^2}+\dfrac{4}{y^2}=5,$
$xy=1.$

17. $15(x^2+y^2)=34xy,$
$\dfrac{1}{x}-\dfrac{1}{y}=2.$

18. $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{257}{16},$
$4(x+y)=17.$

19. $\dfrac{x}{y}+\dfrac{y}{x}=\dfrac{17}{4},$
$x-y=\frac{3}{2}.$

20. $\dfrac{4x}{y}+\dfrac{y}{x}=\dfrac{17}{2},$
$2x+y=20.$

21. $\dfrac{1}{x^3}+\dfrac{1}{y^3}=35,$
$\dfrac{1}{x}+\dfrac{1}{y}=5.$

$\dfrac{1}{x^3}-\dfrac{1}{y^3}=61,$
$\dfrac{1}{x}-\dfrac{1}{y}=1.$

23. $x^3+y^3=351,$
$x^2-xy+y^2=39.$

24. $x^3-y^3=702,$
$x^2+xy+y^2=117.$

25. $8x^3+y^3=2,$
$4x^2-2xy+y^2=1$

26. $8x^3+27y^3=2,$
$4x^2-6xy+9y^2=1.$

***157.** Solve the equations $2x^2y^2 - 13xy + 18 = 0,$ .... (1)

$$x + y = \tfrac{9}{2}. \qquad\qquad (2)$$

Treating (1) as a quadratic for xy,

$$(2xy - 9)(xy - 2) = 0;$$

$$\therefore\ xy = \tfrac{9}{2} \text{ or } 2.$$

The complete solution is then obtained by first solving the equations

$$x + y = \tfrac{9}{2}, \quad xy = \tfrac{9}{2},$$

and then the equations $x + y = \frac{9}{2}$, $xy = 2$, as in Example 2, Art. 155.

***158.** When the variable terms in the equations are *homogeneous*, *i.e. of the same degree*, the following method may be used.

Solve the equations $12x^2 - 4xy + 11y^2 = 64,$ .... (1)

$$16x^2 - 9xy + 11y^2 = 78. \qquad\qquad (2)$$

Eliminate the constant terms, by multiplying across (multiply the left hand side of each equation by the right hand side of the other).

$$78(12x^2 - 4xy + 11y^2) = 64(16x^2 - 9xy + 11y^2),$$

$$39(12x^2 - 4xy + 11y^2) = 32(16x^2 - 9xy + 11y^2).$$

Multiplying out, and re-arranging,

$$77y^2 + 132xy - 44x^2 = 0,$$

$$7y^2 + 12xy - 4x^2 = 0,$$

$$(7y - 2x)(y + 2x) = 0;$$

$$\therefore\ y = \frac{2x}{7} \text{ or } y = -2x.$$

(If the factors cannot be seen, solve as a quadratic for $\dfrac{y}{x}$)

(1) When $y=\frac{2x}{7}$. Substituting this value of y in (1),

$$x^2\left(12-\frac{8}{7}+\frac{44}{49}\right)=64,$$

whence
$$x^2=\frac{49\times 64}{576}=\frac{49}{9},$$
$$x=\pm\frac{7}{3};$$
$$\therefore\ y=\frac{2x}{7}=\pm\frac{2}{3}.$$

(2) When $y=-2x$. Substituting this value in (1),

$$x^2(12+8+44)=64,$$
$$x^2=1,$$
$$x=\pm 1,$$
$$y=-2x=\mp 2;$$

$\therefore$ the reqd. solutions are $\quad x=\pm\frac{7}{3},\ \pm 1,$

$$y=\pm\frac{2}{3},\ \mp 2.$$

***159.** When the above methods are inapplicable, substitution from one equation in the other may be employed.

Solve the equations $\quad 3x^2+4xy+5y^2=31,$ ....(1)

$x+2y=5$ ....(2)

From (2) $x=5-2y$.

Substituting this value of x in (1),

$$3(5-2y)^2+4y(5-2y)+5y^2=31,$$

whence
$$9y^2-40y+44=0,$$
$$(9y-22)(y-2)=0;$$
$$\therefore\ y=\tfrac{22}{9}\text{ or }2,$$
$$x=5-2y=5-\tfrac{44}{9}\text{ or }5-4$$
$$=\tfrac{1}{9}\text{ or }1.$$

*Examples. XXVII. c.

MISCELLANEOUS EXAMPLES IN SIMULTANEOUS QUADRATICS.

Solve the following equations:

1. $x^2+xy=3,$ $y^2+xy=6.$ **2.** $2xy+y^2=16,$ $2x^2-xy=12.$ **3.** $x^2+y^2=xy+7,$ $x^2-y^2=xy-1.$

4. $3x^2 - 5xy = -2,$ $4xy - 3y^2 = 1.$

5. $x^2 - 2xy + 3 = 0,$ $2x + y = 4.$

6. $y^2 + xy = 4,$ $x^2 + 2y^2 - xy = 8.$

7. $x^2 + xy = 3,$ $y^2 + xy = 4.$

8. $6x^2 - 3xy + 11y^2 = 584,$ $x = 5y.$

9. $x^2 + 3xy + 2y^2 = 7,$ $x^2 - y^2 = 4.$

10. $x^2 + xy = 15,$ $xy - y^2 = 2.$

11. $3x^2 + 4xy + 5y^2 = 81,$ $3x = 2y.$

12. $2x^2 + 3xy = 26,$ $3y^2 + 2xy = 39.$

13. $x + y = 6,$ $(x^2 + y^2)(x^3 + y^3) = 1440.$

14. $6x^2 + 3xy - 18y^2 = 20,$ $3x^2 + 6xy = 8.$

15. $x^2 + y^2 = 5,$ $x^2 + xy = 6.$

16. $x^2 = 14 + xy,$ $y^2 = xy - 10.$

17. $x^3 - y^3 = 485,$ $x - y = 5.$

18. $x^2 - 4y = y^2 + 4x = 21.$

19. $\frac{1}{x} - \frac{1}{y} = \frac{1}{12},$ $\frac{4}{x^2} + \frac{6}{y^2} = \frac{5}{12}.$

20. $2x^2 + 3xy + 10 = 0,$ $x^2 + xy - y^2 + 11 = 0.$

21. $3xy + x^2 = 10,$ $5xy - 2x^2 = 2.$

22. $x^2 + xy + y^2 = 61,$ $x + y = 9$

23. $(x+5)(y+7) = (x+27)(y+\frac{5}{7}),$ $xy = 1.$

24. $x^2 + 4y = 28,$ $3x = 4y.$

25. $9x^2 + 6xy - 4y^2 = 1,$ $3x - 2y = -1.$

26. $y^2 - xy = 15,$ $x^2 + xy = 14.$

27. $x^2 + 4y^2 - 3x + y = 67,$ $x - 2y = 1.$

28. $x^2 + xy + y^2 = 49,$ $x^4 + x^2y^2 + y^4 = 931.$

29. $x^2 + xy = 12,$ $xy - 2y^2 = 1.$

30. $2x + 3y = 1\frac{1}{2},$ $4x^2 + 9xy + 9y^2 = 11.$

31. $(x+y)^2 + 3(x-y) = 30,$ $xy + 3(x-y) = 11.$

32. $x^2 + 3xy + y^2 = 1,$ $x^2 - xy + y^2 = 13.$

33. $\frac{y}{x} - \frac{3}{y} - \frac{x+3}{x+4} = \frac{x+y}{xy}.$

34. $x^2 + xy = y^3 - 9x^2y + 64 = 0.$

35. $x^4 - x^2 + y^4 - y^2 = 84,$ $x^2 + x^2y^2 + y^2 = 49$

GRAPHS. (CIRCLES.)

***160. The distance of the point (x, y) from the origin $= \sqrt{(\mathbf{x}^2 + \mathbf{y}^2)}$.**

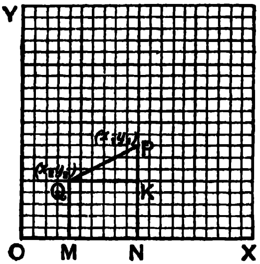
Using this, we may also determine the graph of $y = \sqrt{(25 - x^2)}$ as follows. The equation may be written, $x^2 + y^2 = 25$.

$$\therefore \sqrt{(x^2 + y^2)} = 5.$$

This shows us that the point (x, y) moves at a constant distance of 5 units from the origin.

The graph is therefore a circle, whose centre is at the origin, and whose radius $= 5$.

*161. In the accompanying diagram, let P be the pt. (x_1, y_1) and Q the pt. (x_2, y_2).



Draw PN and QM perp. to the axis of x, and QK perp. to PN.

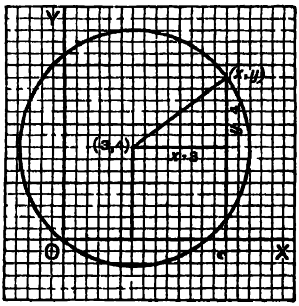
$$PK = y_1 - y_2, \text{ and } QK = x_1 - x_2.$$

$$\therefore\ PQ = \sqrt{(QK^2 + PK^2)} = \sqrt{[(x_1 - x_2)^2 + (y_1 - y_2)^2]}.$$

Thus we see that the distance between the two pts. (x_1, y_1) and (x_2, y_2)

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

*162. *Trace the graph* of $x^2 + y^2 - 6x - 8y = 0$.



This equation may be written $(x-3)^2 + (y-4)^2 = 25$.

$$\therefore\ \sqrt{(x-3)^2 + (y-4)^2} = 5.$$

It is important to notice that if *no constant term* occurs in an equation, the corresponding graph passes through the origin, for by substitution we see that when $x = 0$, one value of y is 0.

The graph of $x^2 + y^2 = 5$ is a circle whose radius is $\sqrt{5}$.

A line $\sqrt{5}$ units long may be drawn either by using Pythagoras' Theorem $(2^2 + 1^2 = 5)$ or by the method of Art. 136.

*Examples. XXVII. d.

Trace the graphs of the following:

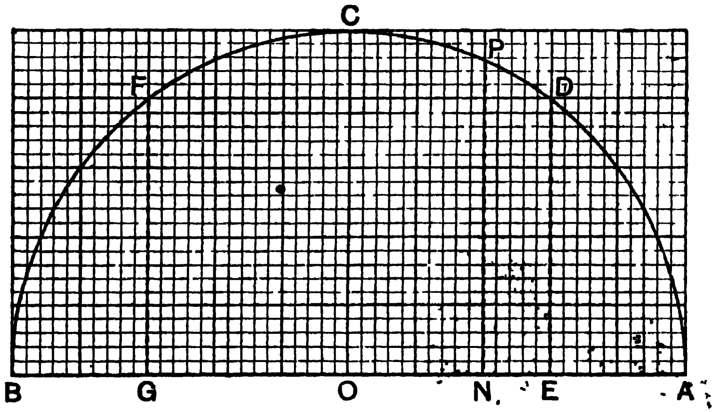
1. $x^2+y^2=36$.
2. $x^2+y^2=0$.
3. $x^2+y^2=49$.
4. $x^2+y^2=81$.
5. $x^2+y^2+8x-8y=0$.
6. $x^2+y^2-8x-6y=0$.
7. $(x-3)^2+(y-4)^2=36$.
8. $(x-1)^2+(y-2)^2=36$.
9. $(x+2)^2+(y-3)^2=25$.
10. $(x-3)^2+(y+3)^2=16$.
11. $\sqrt{(15-2x-x^2)}$.
12. $\sqrt{(21+4x-x^2)}$.
13. $\sqrt{(15+2x-x^2)}$.
14. $\sqrt{(14x-x^2-13)}$.
15. $x^2+y^2=2$.
16. $x^2+y^2=5$.
17. $x^2+y^2=13$.
18. $x^2+y^2=10$.
19. $x^2+y^2=20$.
20. $x^2+y^2=3$.
21. $x^2+y^2+2x+2y=0$.
22. $(x-1)^2+y^2=2$.
23. $(x+2)^2+(y-2)^2=5$.
24. $x^2+y^2+2x+2y=3$.
25. $x^2+y^2-6x+4y+3=0$.
26. $2x^2+2y^2=5$.
27. $2x^2+2y^2-4x+8y+3=0$.
28. $4x^2+4y^2-16x+8y+11=0$.
29. $4x^2+4y^2-24x+11=0$.

GRAPHICAL SOLUTION OF SIMULTANEOUS QUADRATIC EQUATIONS.

***163.** Simultaneous quadratics can often be readily solved by graphical methods.

Example 1. Solve the following equations graphically:

$$x+y=5,\ xy=4.$$



On AB, 5 in. long (the diagram is reduced in printing), describe the semi-circle ACB.

If P is any pt. on the curve and PN is drawn perp. to AB, we know, by Geometry, that $$PN^2=AN.NB.$$

Mark the pts. D, F on the curve where the lengths of the perpendiculars DE, FG on AB are equal to 2 inches ($\sqrt{4}$).

Then $DE^2 = AE \cdot BE$, and $FG^2 = AG \cdot BG$.

$$\therefore \text{ if } AE = x \text{ and } BE = y,$$

$$x + y = AB = 5 \text{ and } xy = AE \cdot BE = DE^2 = 4.$$

$\therefore$ AE, BE are solutions of the given equation.

From the diagram $x = 1$, $y = 4$.

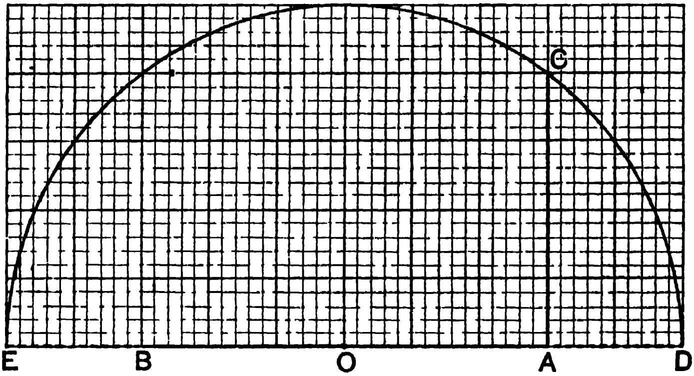
In the same way, AG and BG are solutions, and we have

$$x = 4,\ y = 1.$$

$$\therefore \left.\begin{array}{l} x = 1 \text{ or } 4, \\ y = 4 \text{ or } 1, \end{array}\right\} \text{is the complete solution.}$$

Example 2. Solve the following equations by the graphical method:

$$x - y = 3,\ xy = 4.$$



Take AB 3 in. long and AC at rt. $\angle$s to it 2 in. ($= \sqrt{4}$) long. With O, the mid. pt. of AB as centre, and OC radius, describe the semi-circle ECD, meeting AB produced at D and E.

As in the previous example, $CA^2 = DA \cdot AE$.

$$\therefore \text{ if } AE = x \text{ and } AD = y,$$

$$x - y = AE - AD = AE - BE = AB = 3.$$

Also $xy = EA \cdot AD = AC^2 = 4$.

$\therefore$ AE and AD give a solution of the given equations.

From the diagram see that $x = 4$, $y = 1$.

N.B. $x = -1$, $y = -4$ is also a solution of these equations. The above method does not give negative roots satisfactorily.

The methods of the two preceding examples may be employed to solve some quadratic equations.

Thus to solve $x^2 - 7x + 9 = 0$, we have to factorize the expression $x^2 - 7x + 9$, *i.e.* we have to find two numbers whose sum is 7 and product 9.

We can therefore use the method of Example 1.

In the same way, to solve $x^2 - 3x - 36 = 0$, we have to find two numbers whose difference is 3 and product 36.

We can therefore use the method of Example 2.

***164.** *Solve the following equations graphically:*

$$x^2 + y^2 - 4x - 2y + 1 = 0, \quad 2x - 3y = 3.$$

The first equation may be written

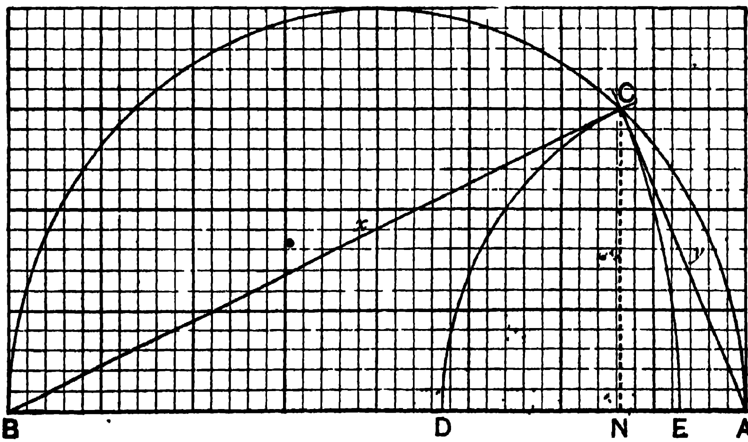
$$(x-2)^2 + (y-1)^2 = 4.$$

Hence its graph is a circle whose centre is at (2, 1) and whose radius is 2.

Draw the circle, and also draw, using the same axes and the same units, the graph of $2x - 3y = 3$, a str. line through the pts. (1·5, 0), (0, −1).

The pts. of intersection of the circle and str line give the roots required.

***165.** *Find approximate solutions of the following equations by a graphical method:* $x^2 + y^2 = 16, \; xy = 6$



The following method depends upon the fact that if ABC is a triangle, right-angled at C, and CN is drawn perp. to the hypotenuse AB, then AC . BC = 2△ABC = CN . AB. Now $\sqrt{16} = 4$, hence on AB, 4 in. long, describe a semi-circle ACB, and take the

pt. C such that the perp. from C on $AB = \frac{6}{4} = 1\frac{1}{2}$ in. (Sqd. paper should be used.)

Then $$AC^2 + BC^2 = AB^2 = 16.$$

Also $$AC \,.\, BC = CN \,.\, AB = \frac{3}{2} \times 4 = 6\,;$$

$\therefore$ AC and BC are roots of the given equation.

With centre A and radius AC describe a circle cutting AB at D

$AC = AD = 1{\cdot}65$ approx. from the diagram.

In the same way $BC = 3{\cdot}65$ approx.;

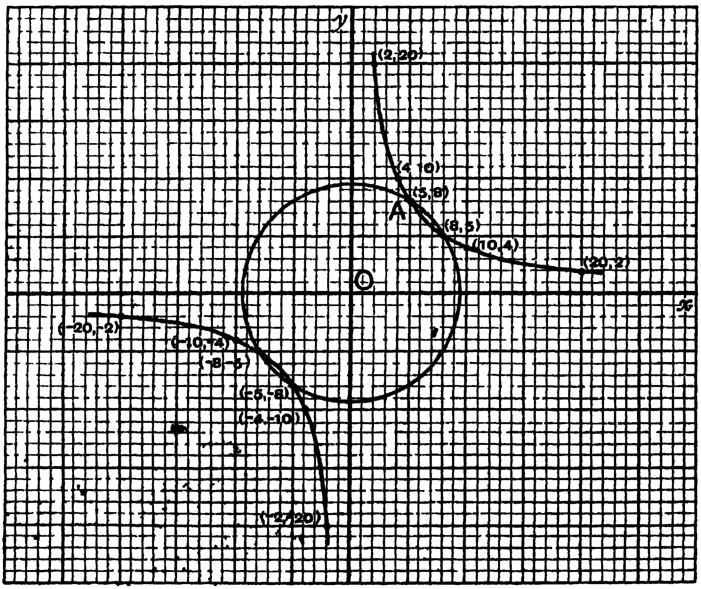
$\therefore$ $1{\cdot}65$, $3{\cdot}65$ are roots of the given equation.

***166.** *To trace the graph of $xy = 40$.*

When

$x = \pm 2$	± 4	± 5	± 8	± 10	± 20	.
$y = \pm 20$	± 10	± 8	± 5	± 4	± 2	

the upper signs being taken together, and the lower signs together.



Plotting these pts. and joining them by an even curve, we have the figure shown in the diagram.

It is observed that the curve lies entirely in the first and third quadrants, and that the two branches are symmetrical in regard to both the axes of co-ordinates.

Hence we have another method of solution of equations of the following type:

$$x^2 + y^2 = 89,$$
$$xy = 40.$$

We first draw the graph of $xy = 40$.

The graph of $x^2 + y^2 = 89$ is a circle whose centre is at the origin, and radius $\sqrt{89}$. Since $89 = 25 + 64 = 5^2 + 8^2$, the length OA in the diagram is the radius. Describing the circle, and reading off the pts. of intersection of the two curves, we have the following solutions:

$$x = 8,\ 5,\ -5,\ -8,$$
$$y = 5,\ 8,\ -8,\ -5.$$

***167.** *Find approximate roots of the equations*

$$xy = 80, \quad x - 2y = 10.$$

From the following table of values, draw the graph of $xy = 80$

$x = \pm 4$	± 5	± 8	± 10	± 20	
$y = \pm 20$	± 16	± 10	± 8	± 4	

Draw the graph of $x - 2y = 10$, a str. line through the pts. (10, 0), (0, −5).

The pts. of intersection of the two graphs give the reqd roots. They will be found to be

$$\left.\begin{aligned} x &= 18{\cdot}6,\ -8{\cdot}6 \\ y &= 4{\cdot}3,\ -9{\cdot}3 \end{aligned}\right\}\text{approx.}$$

Equations of the type of Examples 1 and 2 worked out in this chapter might also be solved by this method.

*Examples. XXVII. e.

Find, approximately, the values of the roots of the following equations, by the use of graphical methods. Verify your results.

(In some cases the exact values of the roots can be obtained.)

1. $x + y = 7,\ xy = 9.$ **2.** $x + y = 9,\ xy = 16.$ **3.** $x - y = 2,\ xy = 16.$

4. $x - y = 4,\ xy = 9.$ **5.** $x + y = 7,\ xy = 5.$ **6.** $x - y = 3,\ xy = 8.$

7. $x^2 - 13x + 36 = 0.$ **8.** $x^2 - 11x + 25 = 0.$ **9.** $x^2 - 8x + 13 = 0.$

Find, approximately, the values of the roots of the following equations, by the use of graphical methods. Verify your results.

10. $x^2-2x-16=0$.
11. $x^2+y^2=4$, $2x-y=1$.
12. $x^2+y^2=8$, $x+2y=2$.
13. $x^2+y^2-2x-4y+1=0$, $5y-5x=3$.
14. $4x^2+4y^2+8x-4y=11$, $x=2-2y$.
15. $x^2+y^2=9$, $4x+3y+6=0$.
16. $x^2+y^2=36$, $xy=15$.
17. $x^2+y^2=225$, $xy=80$.
18. $xy=80$, $2x-y=10$.

CHAPTER XXVIII.

FURTHER EXAMPLES ON SYMBOLICAL REPRESENTATION.

Examples. XXVIII.

1. A man rows x miles an hour in still water, and the current runs at the rate of y miles an hour:

(i) How many miles an hour does the man row with the current?
(ii) .... against ....?
(iii) How long does he take to row a miles with the current?
(iv) .... against .... ?

2. Money is invested at simple interest at the rate of x per cent. per annum:

(i) What is the interest on 1£ for a year?
(ii) .... 1£ .. y years?
(iii) ... . z£ ... . ?
(iv) What does z£ amount to in .... ?

3. Calculating simple interest at the rate of x per cent. per annum,

(i) What is the present value of 100£ due in one year?
(ii) .... a£ ....?
(iii) ... . 100£ .... y years?
(iv) .... a£ ....?

4. A train runs at the rate of y miles an hour:

(i) How long does it take to do one mile?
(ii) .... z miles?
(iii) .... z miles at the above rate, and another z miles at double the rate?
(iv) How many miles does it run in a hours at the slower rate?

5. A can do a piece of work in x hours, B can do it in y hours:

(i) What fraction of the work do A and B do, working together, in one hour?
(ii) .... a hours?
(iii) How long do they take to do the work when working together?
(iv) .... three-quarters ....?

6. One pipe, running alone, fills a cistern in x hours; a second, running alone, fills it in y hours; and a third, also running alone, empties it in z hours:

(i) What fraction of the cistern do they fill, all running together, in an hour?

(ii) How long do they take to fill the cistern, all running together?

7. x£ is the simple interest on y£ for z years:

(i) What is the simple interest on y£ for one year?

(ii) .... 1£ ....?

(iii) ....100£ ....?

(iv) .... a£ ... b years?

8. In x years y£ amounts to z£ at simple interest:

(i) What is the interest on y£ for x years?

(ii) .... y£ ... one year?

(iii) .... 1£ .... ?

(iv) .... a£ ... b years?

(v) What is the rate of interest?

9. Apples cost x pence per dozen:

(i) What does a man give for one apple?

(ii) .... he .... y apples?

(iii) What does he give for one apple when the price is raised a penny per dozen?

(iv) What does he give for y apples at the higher price?

(v) How much do a apples cost at the cheaper price?

(vi) .... higher ...... ?

10. A man invests money at compound interest at the rate of x per cent. per annum:

(i) What is the interest on 1£ for one year?

(ii) .... amount of 1£ .... ?

(iii) .... a£ .... ?

(iv) .... interest on .... ?

(v) .... amount of 1£ ... 2 years?

(vi) .... 3 ...... ?

(vii) .... n ...... ?

(viii) .... P£ ... 2 years?

(ix) .... 3 ...... ?

(x) .... n ...... ?

(xi) .... interest on .... ?

11. If simple interest is calculated at the rate of x per cent. per annum,

(i) What is the discount on 100£ due in one year?

(ii) .... a£ ....?

(iii) .... 100£ .... y years?

(iv) .... a£ ....?

12. A man can do a piece of work in x hours; a woman does half as much as a man, and a boy half as much as a woman. What fraction of the work will

(i) A man, a woman, and a boy together do in 1 hour?

(ii) 2 men, 3 women, and 4 boys ....?

13. One man walks x miles an hour, and another y miles an hour starting at the same time, in the same direction.

(i) How much apart are they in an hour if the first man is the quicker walker?

(ii) How much apart are they in a hours?

(iii) How long does the first take to gain one mile on the other?

(iv) .... b miles ....?

Express the following in the form of equations:

14. The product of two consecutive numbers of which x is the smaller is less than the product of the next higher two consecutive numbers by y.

15. A man bought a cows at x£ each, and b sheep at y£ each, and altogether spent z shillings.

16. Apples are sold at x pence a dozen, and pears at y pence for 10. a apples and b pears cost z shillings.

17. x men form a hollow square, four ranks deep, with y men on each outside face of the square.

18. A hollow square is formed by a men, y ranks deep, with z men on each outside face of the square.

19. A fraction whose numerator is x, and denominator y, is increased by a when the numerator is increased by b, and the denominator decreased by c.

20. x dozen of wine at a shillings a dozen, and y dozen at b shillings a dozen, cost c shillings a dozen on the average.

21. The area of a room x ft. long and y ft. wide is doubled when its length and breadth are each increased by a feet.

22. In travelling a yards, the fore wheel of a carriage makes n revolutions more than the hind wheel. Take x feet for the circumference of the fore wheel and y feet for that of the hind wheel.

23. One pipe will fill a cistern in x hours, a second will fill it in y hours; running together they fill it in z hours.

24. A starts off on a journey at x miles an hour; and n hours afterwards, B starts off at y miles an hour, and catches A up in a hours from A's start.

25. Two men start simultaneously to walk from A and B to B and A respectively, a distance of n miles. They walk at x miles an hour and y miles an hour, and meet in a hours.

26. Form the equation for the above problem when the second man starts b hours after the first, and they meet a hours after the first man started.

27. Between two places one mile apart there are x telegraph posts in a straight line, y yards apart.

28. Between two places a miles apart, there are x telegraph posts in a straight line, y yards apart.

29. A man spends one-third of his income of x£ in board and lodging, one-fifth in dress and one-tenth in sundries, and has y£ left at the end of the year.

30. A tradesman makes in a year a profit of x per cent. on his capital of y£ and has z£ at the end of the year.

31. A man gains x per cent. on a£ and loses y per cent. on b£, and altogether makes a profit of c£.

32. A man runs a miles at x miles an hour, b miles at y miles an hour, and c miles at z miles an hour, and takes d hours over the whole journey.

33. A man is hired for x days. He is paid y shillings a day for a days, and is fined z shillings a day for the rest of the time because he absents himself. He receives c£.

CHAPTER XXIX.

PROBLEMS INVOLVING QUADRATIC EQUATIONS.

168. **Example 1.** A number of two digits is less than four times the product of its digits by 11, and the digit in the tens' place exceeds the digit in the units' place by four. Find the number.

Let x be the digit in the units' place.

Then $x+4$ is the digit in the tens' place.

The number $=10(x+4)+x=11x+40$.

Four times the product of its digits $=4x(x+4)$;

$$\therefore\ 4x(x+4)-(11x+40)=11,$$
$$4x^2+16x-11x-40=11,$$
$$4x^2+5x-51=0,$$
$$(x-3)(4x+17)=0,$$
$$x=3 \text{ or } -\frac{17}{4}.$$

$\therefore$ 3 is the digit in the units' place, and $3+4\,(=7)$ the digit in the tens' place.

73 is therefore the reqd. number.

The solution $-\frac{17}{4}$ is inadmissible, because the digits of a number are *positive integers.*

Example 2. A reduction of 2 pence a dozen in the price of eggs will give 6 more for three shillings and sixpence: find the price per dozen.

Let x pence be the price of 12 eggs.

For 42 pence we obtain $\frac{12}{x}\times 42$ eggs.

When $x-2$ pence is the price of 12 eggs, we obtain $\frac{12}{x-2}\times 42$ for 3*s.* 6*d.*

$$\therefore\ \frac{12}{x-2}\times 42-\frac{12}{x}\times 42=6,$$

$$\frac{84}{x-2}-\frac{84}{x}=1,$$

$$84x-84(x-2)=x^2-2x,$$

$$x^2-2x-168=0,$$

$$(x-14)(x+12)=0,$$

$$x=14 \text{ or } -12.$$

$\therefore$ 14 pence a dozen is the reqd. price.

Example 3. A train does a journey of 240 miles at a uniform rate; if it had travelled 4 miles an hour slower, it would have taken 2 hours more over the journey: find its rate of travelling.

Let x miles an hour be the reqd. rate of travelling.

At the higher speed, the train took $\frac{240}{x}$ hours over the journey.

At the slower speed, $x-4$ miles an hour, it took $\frac{240}{x-4}$ hrs. over the journey.

$$\therefore \text{ by hypothesis, } \frac{240}{x}=\frac{240}{x-4}-2.$$

Multiplying up,
$$240(x-4)=240x-2x(x-4),$$
$$2x^2-8x-960=0,$$
$$x^2-4x-480=0,$$
$$(x-24)(x+20)=0;$$
$$\therefore x=24 \text{ or } -20.$$

$\therefore$ the train travels at the rate of 24 miles an hour, the negative solution being inadmissible.

It will be proved later on that every quadratic equation has two roots. As a consequence of this, inadmissible solutions of problems involving quadratic equations will often occur In this case, the negative solution would imply that the train travelled *backwards* at 20 miles an hour.

Example 4. A man invests his money at compound interest for two years at a certain rate per cent. and finds that he receives 5 shillings per cent. more than if he had invested it at simple interest. Find the rate per cent.

Let x be the rate per cent.

At compound interest, 100£ amounts to $(100+x)$£ in the first year.

The interest on $(100+x)$£ for the second year $=(100+x)\times\frac{x}{100}$.

$$\therefore \text{ the interest on £100 for the two years} = x+\frac{(100+x)x}{100}.$$

At simple interest, the interest on 100£ for the two years $=2x$.

$$\therefore x+\frac{(100+x)x}{100}=2x+\tfrac{1}{4},$$

whence
$$x^2=25,$$

and
$$x=\pm 5.$$

$\therefore$ 5 per cent. is the reqd. rate of interest.

Example 5. Two pipes running together will fill a cistern in $6\frac{2}{3}$ minutes. If one pipe, running alone, took a minute less to fill the cistern, and the other pipe, running alone, took 2 minutes more to do the same, then the two, running together, would fill the cistern in 7 minutes. Find in what time the cistern will be filled by each pipe running alone.

Let the first pipe, when running alone, fill the cistern in x minutes, and let the second pipe .... y ....

When running alone, the first pipe fills $\frac{1}{x}$ of the cistern in one minute

.............................second .... $\frac{1}{y}$ ....

But since by hypothesis they running together fill the cistern in $\frac{20}{3}$ min.

$\therefore$ in one minute .... $\frac{3}{20}$ of the cistern;

$$\therefore \frac{1}{x}+\frac{1}{y}=\frac{3}{20}. \qquad (1)$$

In the second case, the first pipe fills the cistern in $x-1$ min.

.............................second .... $y+2$ ...

$$\therefore \frac{1}{x-1}+\frac{1}{y+2}=\frac{1}{7}. \qquad (2)$$

From (1), $$\frac{1}{x}=\frac{3}{20}-\frac{1}{y}=\frac{3y-20}{20y}.$$

$$\therefore x=\frac{20y}{3y-20}. \qquad (3)$$

From (2), $$\frac{1}{x-1}=\frac{1}{7}-\frac{1}{y+2}=\frac{y-5}{7(y+2)},$$

$$x-1=\frac{7(y+2)}{y-5}. \qquad (4)$$

From (3) and (4), $$\frac{20y}{3y-20}-1=\frac{7(y+2)}{y-5}.$$

From this quadratic for y, $y=12$ will be found to be the only admissible solution.

Substituting in (3), $x=15$.

$\therefore$ the pipes would fill the cistern in 15 and 12 minutes respectively.

Examples. XXIX. a.

1. The difference of two numbers is 2, and the sum of their squares is 244: find them.

2. A room is 4 feet more in length than in breadth, and its area is 192 sq. ft.: find its dimensions.

3. The product of two consecutive even numbers is 288. What are they?

4. Find two consecutive numbers such that the sum of their squares is 481.

5. x yards of cloth at $x-3$ shillings per yard were bought for 13*s.* 9*d.* What was x?

6. What number when increased by 30 will be less by 12 than its square?

7. Find the number which, added to its square root, will make 182.

8. The length of a rectangular field is twice its breadth. If 20 yds. were added to its length and 30 to its breadth, its area would be 10,458 sq. yds. Find the dimensions of the field.

9. In a right-angled triangle one of the sides containing the right angle is 3 feet in length, and the square on the hypotenuse is 4 times the area of the triangle. Find the length of the remaining side.

10. A man bought x oxen for £120. Another bought 3 more for the same money. What was the cost of an ox to the first man, what to the second? If the difference was £2 per ox, what were the numbers bought?

11. A rectangular table 9 ft. by 6 ft. has a rectangular table-cloth which hangs down to the same depth at the ends and sides. What is that depth if the area of the cloth is twice that of the table?

12. The product of two numbers which differ by 3 is 40: find them.

13. When 13 times a certain number is subtracted from the square of the number, the result is 30. Find the number.

14. A motor-car does a journey of 192 miles at the average rate of x miles per hour, and a second car does the same journey at the average rate of $x+4$ miles per hour. How long does each car take over its journey?

If the difference of these times is 4 hours, find the value of x.

15. The difference of two numbers is 3, and the sum of their squares is 117. Find the numbers.

16. A man rents x acres of land for £54 per annum. How much does he pay per acre? If he sublets all except 8 acres at 5*s.* per acre more than this and receives £64 per annum, find the value of x.

17. A rectangular enclosure has an area of 2000 sq. yds., and its perimeter is 180 yds. in length. Find the lengths of its sides.

18. A man rows 6 miles down stream at x miles per hour, and the same distance up stream at $x-1$ miles per hour. How long does he take over each journey? If he takes $3\frac{1}{2}$ hours over the two journeys, find the value of x.

19. If the hind wheel of a carriage is x ft. in circumference, how many revolutions does it make in a mile? If the front wheel is 2 ft. smaller in circumference, and makes 24 more revolutions in a mile than the hind wheel, find the value of x.

20. A train travelling at x miles an hour for $x+12$ minutes goes 21 miles. Find x.

21. A bill of 80 shillings was shared equally between x persons. What did each pay? If two were excused, what would each pay? If this made a difference of 2 shillings to each, what was x?

22. 110 bushels of coals are equally divided among x poor persons. What number of bushels does each receive? If this number is one less than the number of persons, how many are there?

23. Two trains each run a distance of 330 miles, one at x miles per hour, the other at $x+5$. The faster takes half an hour less than the other for the whole distance. What are their speeds?

24. A can do a piece of work in x days, B in $x+12$ days. What fraction of the work can they respectively do in a day? If together they take 8 days, what times will they take separately?

25. A cistern can be filled by two pipes in $1\frac{1}{3}$ hours. The larger pipe by itself will fill the cistern sooner than the smaller by 2 hours. Find what time each will take separately.

26. A car travels 15 miles an hour faster downhill than uphill, and takes $2\frac{1}{10}$ minutes to run up and down a hill one quarter of a mile long, when the time taken in turning is deducted. Find its speed downhill.

27. A fraction, whose numerator is less than its denominator by 3, is doubled if 6 is added to the numerator and 5 to its denominator. Find its value.

28. The product of the two highest of five consecutive integers exceeds twice the product of the two smallest by 6. Find them.

29. The tens digit of a certain number is the square of a number which is 2 less than the units digit, and the sum of the two digits is 14. Find the number.

30. A rectangle whose area is 54 sq. ft. has its sides resp diminished by 5 feet and 2 feet and so becomes a square. Find the length of a side of the square.

31. A train does a journey of 288 miles at a certain average speed and is one hour late. If it had travelled 4 miles per hour faster it would have been punctual. Find its speed.

32. A point travels for 8 secs. at the rate of x feet per sec., and then for $4x$ secs. at the same rate. If the total space described is 96 feet, find the value of x.

* Examples. XXIX. b.

1. Find two numbers whose difference is 2, such that twice the square of the less shall exceed the square of the greater by unity.

2. The plate of a looking glass is 18 inches by 12 inches. It is to be framed with a frame of uniform width, the area of which is to be equal to that of the glass. Find the width of the frame.

3. Mr. Gladstone was born in the year A.D. 1809. In the year A.D. x^2 he was $x-3$ years old: find x.

4. When 17 times a certain number is subtracted from twice its square, the remainder is 84: find the number.

5. The tens digit of a certain number is the square of the units digit, and the sum of its two digits is 12: find the number.

6. A man runs 600 yards at a certain pace, and then doubling his pace, does another 600 yards. If he took $2\frac{1}{2}$ minutes over the 1200 yards, find the pace he started at, in yards per second.

7. Find two numbers whose difference is 3, and the sum of whose squares is 317.

8. A's rate of travelling is one mile an hour less than B's, and B can go 21 miles in 20 minutes less than it takes A to go 20 miles. How many miles an hour can A travel?

9. Find a number which together with its square amounts to 56.

10. Two trains each run a distance of 330 miles. One of them, whose average speed exceeds that of the other by 5 miles an hour, takes half-an-hour less to travel the whole distance. Find their average speeds.

11. A lady bought 28 yards of linen and a certain length of silk. The whole cost was 65s., the silk cost as many shillings per yard as there were yards of it, and 8 times as much as the same number of yards of linen. Find the price of the silk per yard.

12. P rides from A to B in one hour at a uniform speed. Q rides for one-third of the way 2 miles an hour faster than P, and for the rest of the journey 1 mile an hour slower than P, thus taking 40 seconds longer. Find the distance from A to B.

13. A person rents some land for £48. He cultivates 8 acres himself, and sub-letting the rest for 15s. per acre more than he pays, receives in rent £54 per annum. Find the number of acres.

14. One side of a room is 6 ft. longer than the other, and 924 square feet of paper are required to cover its walls. Now if the room were 3 feet higher, the same amount of paper would be required to cover three of its walls, one of the shorter walls being left uncovered. Find the dimensions of the room.

15. Of two square courtyards one contains as many square yards as it costs shillings to pave the other, and a side of the second contains as many linear yards as it costs pounds to pave the first, also the length of a side of the first exceeds that of the second by 3 yards, and the cost of paving the first exceeds that of paving the second by £2. Find the sizes of the courtyards, and the costs of paving.

16. Ten minutes after the departure of an express train a slow train is started, travelling on the average 20 miles less per hour, which reaches a station 250 miles distant $3\frac{1}{2}$ hours after the arrival of the express. Find the rate at which each train travels.

17. The length of a room is 2 feet more than its breadth, and its height is three-quarters of its breadth. If the area of the ceiling be 42 square feet more than that of the longer side, find the dimensions of the room.

18. A bicyclist, having ridden 72 miles and stopped an hour on the way, finds that, if he had ridden faster by one mile an hour and stopped two hours on the way, he would have accomplished the journey in the same time. At what pace did he ride?

19. In 100 minutes a boat's crew row $3\frac{1}{2}$ miles down a river and back again. If the river runs at 2 miles an hour, what is the pace of the boat in still water?

20. In going a quarter of a mile along a straight road the hind wheel of a bicycle turns 11 times more than the front wheel. Had the front wheel been 3 inches longer in circumference than it actually is, the hind wheel

would have turned 16 times more than the front wheel. Find the circumference of each wheel.

21. A battalion of soldiers when formed into a solid square present sixteen men fewer in the front than they do when formed into a hollow square four deep. Find the number of men.

22. A man buys pigs, geese, and ducks. If each of the geese had cost a shilling less, one pig would have been worth as many geese as each goose is actually worth shillings. A goose is worth as much as two ducks, and 14 ducks are worth seven shillings more than a pig. Find the price of a pig, a goose, and a duck respectively.

23. A sum of money is divided among A, B, and C, so that a third of the whole sum exceeds A's share as much as B's exceeds a quarter of the whole. What part does C get?

24. A cyclist rides 3 miles an hour faster downhill than uphill; and takes the same time to ride 22 miles downhill and 48 miles uphill that he takes to ride 50 miles downhill and 27 miles uphill. What is his speed uphill?

25. A carrier charges 3*d.* each for all parcels not exceeding a certain weight; and on heavier parcels he makes an additional charge for every 7 lbs. above that weight. The charge for half a cwt. is 1*s.* 3*d.*, and the charge for 9 stones is five times that for 1 qr. What is the scale of charges?

26. A boat's crew row a certain distance against the stream in $8\frac{4}{7}$ minutes. If there were no current they would row the distance in 7 minutes less than it takes them to drift the distance down the stream. In what time would they row the course down the stream?

27. A man being asked his age, answered, 'If you multiply my two digits together, the number formed will be my age 22 years ago, and if you add all the digits of the two ages you will have one-third of my present age.' How old is he?

28. Three travellers A, B, C make the same journey. A's rate of travelling is 3 miles an hour greater than B's, and B's rate is 2 miles an hour greater than C's. A accomplishes the journey in 3 hours less time than B, and B in 4 hours less time than C. Find the rate of each, and the length of the journey.

29. A giant weighs 3 lbs. for every inch of his height, and the square of his height in feet exceeds his weight in stones by 31. Find his height and weight.

30. A labourer undertakes to carry a load a certain distance, agreeing to take one shilling for each cwt. moved one mile. He earns 4·05£, and the distance in miles exceeds the number of cwts. carried by 4·05. Find the load and the distance.

31. A rectangular enclosure is half an acre in area, and its perimeter is 201 yards. Find the lengths of its sides.

32. The sum of two numbers is six times their difference, and their product exceeds twice their sum by 11. Find the numbers.

33. If the longer side of a rectangle be increased by 3 yards, and the shorter by 2 yards, one side becomes double the other, and the area is doubled. Find the lengths of the sides.

34. A lawn, rectangular in shape, contains 864 square yards; if it were 4 yards longer and 3 yards narrower its area would be the same. Find its dimensions.

35. The circumference of one wheel is 8 inches longer than that of another, and the first makes 72 fewer revolutions in a mile: find the circumference of each.

36. A slow train takes 5 hours longer in journeying between two given termini than an express, and the two trains when started at the same time, one from each terminus, meet 6 hours afterwards. Find how long each takes in travelling the whole journey.

37. The area of a rectangular room is 328 square feet, and its perimeter is 73 feet: find the lengths of its sides.

38. A boat's crew finds that the number of minutes which they just require to row 4 miles in a river against the stream exceeds by 31 the number of miles per hour they can row in still water; while it takes them 20 minutes to row the 4 miles with the stream. Find the rate at which the river flows.

39. In a mixed number the integer is 98 times the fraction. The numerator of the fraction being unity, and its denominator less by 7 than the integer, find the mixed number.

40. Two men start simultaneously from opposite ends of a road and meet at the end of 6 minutes. They pass one another, and each continuing to the end from which the other started, one ends his walk 5 minutes before the other. How long does each take?

41. A, B, and C walk from P to Q, a distance of 30 miles; A starts $2\frac{1}{2}$ hours before B, and B $1\frac{1}{4}$ hours before C, and they arrive at Q together. If B had started half-an-hour earlier, he would have passed A 2 hours before A reached Q. Find the rates at which A, B, and C walk.

42. A grocer has two weights, one as much over a lb. as the other is under a lb., and he finds that on selling 511 lbs. 14 ozs. of tea at 2s. 6d. a pound he gets £2 more by using the lighter weight than he would have done by using the heavier: what were the respective weights?

43. A gentleman arrives at the railway station nearest to his house an hour and a half before the time at which he had ordered his carriage to meet him. He sets out at once to walk at the rate of 4 miles an hour, and meeting his carriage when it had travelled 2 miles, reaches home exactly an hour earlier than he had originally expected. How far is his house from the station, and at what rate was his carriage driven?

44. The figures which express the pounds and the pence in a certain sum of money will change places if £2 19s. 9d. be added to it, and those which express the shillings and the pence would be interchanged by subtracting 2s. 9d. What alteration would be produced in the sum of money by interchanging the figures which express the pounds and shillings?

45. Two cyclists travel, one from A to B, the other from B to A, by the same road, and at uniform speeds. They start at the same moment. One reaches B $2\frac{1}{2}$ hours, the other reaches A 3 hours 36 minutes after they meet. How long was each on the journey?

46. A and B walk from one town to another. After walking 6 miles at a uniform speed A arrives at the top of a slope where he mends his pace by 1 mile an hour. B starts forty minutes later, and, after walking at a uniform speed, reaches the slope 10 minutes later than A: here increasing his speed by $\frac{1}{2}$ a mile an hour, he overtakes A just as the town is reached. A would have covered the distance in half an hour less, had he walked the whole distance with B's initial speed. Find the distance and the speeds.

47. Two towns A, B are connected by two roads, one of which is twice as long as the other. A man walked by the shorter road from A to B, and returning immediately by the longer road met one mile from B another man who started at the same time from A on a tricycle and travelled 3 miles an hour faster; and when he had walked 2 hours longer he again met the tricyclist who had passed through B and A without stopping. Find the lengths of the two roads, and the rate at which each man travelled.

48. What fraction will be increased by $\frac{1}{38}$ when unity is added to both numerator and denominator, and diminished by $\frac{8}{57}$ when 4 is subtracted from each of them?

49. A railway passenger observes the time of transit over three successive miles, and finds that the time for the first mile exceeds the time for the second by twice as much as the time for the second exceeds the time for the third. He also calculates that the average speed for the train in the first mile is 5 miles per hour less than in the second, and 8 miles per hour less than in the third. Find the time of traversing each of the three miles.

50. A cask A, of 20 gallons capacity, is filled with brandy, a certain quantity of which is afterwards drawn off into an equal cask B, which is then filled up with water. After this, A is filled up with some of the mixture in B; and when $6\frac{2}{3}$ gallons of the mixture now in A is poured back into B, the two casks contain equal quantities of brandy. How much was at first taken out of A?

CHAPTER XXX

EXAMPLES FOR REVISION.

XXX. a. (*Oral.*)

Read off the square root of

1. $\cdot 25a^6b^2$. **2.** $\cdot 0001\frac{x^6}{y^2}$. **3.** $\frac{2\cdot 5}{10}x^4y^2$. **4.** $\frac{x^{10}}{\cdot 0064}$.

5. $4a^2-8ab+4b^2$. **6.** $\frac{1}{x^2-6x+9}$. **7.** $4x^2\pm 12xy+9y^2$.

8. $1\pm 4a^2b+4a^4b^2$. **9.** $x^2\pm 2+\frac{1}{x^2}$. **10.** $x^2\pm\frac{5ax}{2}+\frac{25a^2}{16}$.

11. $1\pm 2(a-b)+(a-b)^2$. **12.** $\left(\frac{a}{b}-2\right)^2+4\left(\frac{a}{b}-2\right)+4$.

13. $(x+5y)^2-10y(x+5y)+25y^2$. **14.** $(a+b)^2+2(a^2-b^2)+(a-b)^2$.

15. $4x^4\pm 2+\frac{1}{4x^4}$. **16.** $4x^4\pm 4+\frac{1}{x^4}$.

Read off the roots of the following quadratic equations:

17. $x^2-9x+20=0$. **18.** $x(x+3)=x+3$. **19.** $(x-4)(x-5)+2(x-5)=0$.

20. $(x^2-16)+(x-4)=0$. **21.** $x^2+5x=0$. **22.** $25x^2-16=0$.

23. $x(2x+1)-\frac{1}{2}(2x+1)=0$. **24.** $3x(4x-5)=7(4x-5)$.

Read off the roots of the following quadratic equations:

25. $3x(2x-3)+\frac{1}{3}(2x-3)=0.$

26. $3(x-a)+x(x-a)=0.$

27. $x-2+\frac{1}{x}=0.$

28. $7(5x-7)=\frac{3x}{2}(5x-7).$

29. $(x-1)^2=9.$

30. $x+2+\frac{1}{x}=0.$

31. $2x-2+x(x-1)=0.$

Find, by inspection, *one root* in each of the following equations:

32. $2x-2+(7x-3)(x-1)=0.$

33. $\frac{2x-3}{7}+\frac{27x}{17}(6x-9)=0.$

34. $\frac{13x}{11}(2x-1)-5(x-\frac{1}{2})=0.$

35. $7(3x-6)+11x(2x-4)-21x(5x-10)=0.$

36. $\frac{3x}{7}\left(3x-\frac{3}{2}\right)+(11x+14)\left(7x-\frac{7}{2}\right)=0.$

37. $\frac{5x-1}{x-7}+\frac{2x-\frac{2}{5}}{x+3}=0.$

XXX. b.

1. Simplify $\frac{a}{2x+3a}-\frac{a}{3a-2x}-\frac{4ax}{8x^2-18a^2}.$

Deduce the solution of the equation formed by equating the expression to zero. Test your result.

2. Write down (*a*) the square root of $(a+b)^2-2(a+b)+1$,
(*b*) the square of $a+b-c$,
(*c*) the cube of $a+b$.

3. Solve the equation $4x+\frac{3}{x-1}+4=0.$ Test your answer.

4. Draw enough of the graph of $y=x^2$ to determine $\sqrt{8}$ and $\sqrt{13}$. Use one inch as x unit and one-tenth of an inch as y unit.

5. Solve the equations $3x-7y=2$, $xy=3$.

6. Use the remainder theorem to prove that $x-a+b$ is a factor of
$$(x-a)^2+(2b-c)(x-a)+b^2-bc.$$

7. Find a fraction which becomes equal to $\frac{1}{2}$ if the numerator is increased by 2, and equal to $\frac{1}{3}$ if its denominator is increased by 3.

XXX. c.

1. Simplify $\frac{1}{x^2-ax+bx-ab}+\frac{1}{x^2-ax-bx+ab}.$ Check your result.

2. Determine values of a which will make $x^2-ax+25$ a complete square.

3. Solve the quadratic $x-4=1-\frac{14}{x+4}.$ Check your result.

4. Find the square root of $25x^4-70x^3+89x^2-56x+16$.

5. Draw the graph of $y=5x-x^2$. From your figure determine the value of x which gives $5x-x^2$ a maximum value. What is the value of y in this case? Test your results algebraically.

6. Solve the equations $x^2+y^2=25$, $x+y=7$ graphically and by algebra.

7. Between one census and the next the native population of a town increased by 8 per cent., while the number of foreigners decreased from 200 to 150. The increase in the total population was 7 per cent. What was the total population at the second census?

XXX. d.

1. Simplify $\dfrac{2a}{a+2b}+\dfrac{3a}{a-3b}+\dfrac{8a^2}{(6b-2a)(a+2b)}$.

2. Write down (i) the square root of $(x^2-x)^2-8(x^2-x)+16$.
(ii) the square of $a-2b+c$.
(iii) the cube of $a+2b$.

3. Using half an inch as x unit, and one-tenth of an inch as y unit, draw the graph of $y=x^2-3x+2$, for integral values of x, from -2 to 5. What do you deduce as to the equation $x^2-3x+2=0$? Give reasons.

4. Draw enough of the graph $y=x^2$ to determine the square roots of 54·8 and 58·5, correct to two decimal places. Use a large x unit.

5. Solve the equations $\dfrac{2}{x}-\dfrac{1}{y}=\dfrac{5}{12}$, $xy=12$.

6. Find the values of a which will make the expression

$$8x^3+a^2x^2-10ax-48$$

exactly divisible by $x-2$.

7. A clock is two minutes slow but is gaining. If it were three minutes slow, but were gaining half a minute a day more than it does, it would show correct time exactly 24 hours sooner. How much does the clock gain in a day?

XXX. e.

1. Simplify $\dfrac{2-x}{3-2x-x^2}-\dfrac{x-3}{x^2+x-2}$.

2. What values of a will make $9x^2+axy+4y^2$ a complete square?

3. Solve the quadratic $6(x^2-2)=x$, by completing squares, and verify your results by means of the formula for solving quadratic equations.

4. Determine graphically between what values of x the expression $35-4x-4x^2$ is positive. Verify your result by algebra.

5. Solve the equations $3x^2+4xy=11$,
$4y^2+3xy=22$.

6. Find the square root of $16x^4-16x^3+4x^2+8x-4+\dfrac{1}{x^2}$.

7. A sum of money is distributed among some children, each child receiving the same amount. If a shilling less had been given to each, 36 more children could have participated; and if a shilling more had been given to each, the number of children would have had to be reduced by 20. Find the sum distributed.

XXX. f.

1. Simplify $\dfrac{6x^2+x-1}{2x^2-5x-12}\times\dfrac{6x^2+11x+3}{9x^2-1}\div\dfrac{2x^2+9x+4}{x^2-16}$.

2. Prove that $x-a$ is a factor of $x^3-(a+b-c)x^2+(ab-bc-ca)x+abc$.

3. Solve, graphically, the equation $2x^2+x-13=0$. Get your results correct to one decimal place, and check your answer.

4. Find the maximum value of $7x-x^2$, and the minimum value of x^2-5x.

5. Solve the equations $x^2-5xy-14y^2=10$,
$x-7y=1$.

6. If $a^2=b^2+c^2$, prove that $(a+b+c)(b+c-a)(a+c-b)(a+b-c)=4b^2c^2$.

7. A fruiterer sold a certain quantity of oranges for £6. 10*s*. If he had given two more oranges for a shilling, the same quantity would only have realized £5. 17*s*. How many oranges did he sell?

XXX. g.

1. Simplify $\dfrac{x^4+2x^2y^2+y^4}{x^4+x^2y^2+y^4}\times\dfrac{x^6-y^6}{x^4+x^2y^2}\div\left(1-\dfrac{y^4}{x^4}\right)$.

2. Prove that $(a-b)$, $(b-c)$, $(c-a)$ are factors of
$$a^4(b-c)+b^4(c-a)+c^4(a-b).$$

3. Solve the equation $4x^2-3x-12=0$ graphically and by algebra.

4. Use a geometrical method to find the value of $\sqrt{8}$.

5. Solve the equations $(x+2y)^2-3(x+2y)-28=0$,
$$x-2y=5.$$

6. Extract the square root of $x^4+1-12x(x^2+1)+38x^2$.

7. A man starts at 2 p.m. to walk to a place 13 miles off. He walks at a uniform speed till 4 p.m., when he increases his speed by one mile an hour, and reaches his destination at 5.30 p.m. At what speed did he walk during the first two hours?

XXX. h.

1. Resolve into factors: (i) x^4-3x^2+9,
(ii) $512(x-\frac{1}{8})^3-(8ax-a)^3$.

2. Simplify $$\frac{(a+b)x}{(x+a)(x-b)}+\frac{(b+c)x}{(x+c)(b-x)}.$$

3. Divide $(x^2-y^2)^3-z^6$ by $x^2-y^2-z^2$.

4. A certain port wine is worth 47*s*. a dozen now, and increases in value at the rate of 3*s*. a dozen per annum. Draw a graph to determine its worth in coming years, and read off its value per dozen in 7, 13, and 17 years.

5. Solve the equation $5x^2-5x-21=0$ graphically and by algebra, getting your results correct to one decimal place.

6. Solve the equations $x^2+y^2+1=3xy$,
$$2(xy+4)=3y^2.$$

7. One-fourth of the subscribers to a certain school gave a sovereign apiece, one-fourth of the remainder gave half-a-sovereign apiece, and the rest each gave a florin. If the three sets of subscribers raised their subscriptions to a guinea, half-a-guinea, and half-a-crown respectively, the total increase in the subscriptions would be £2. 10*s*. 0*d*. How many subscribers were there and what was the total amount subscribed?

XXX. k.

1. Multiply $8a^5-12a^4b-54a^2b^3+243b^5$ by $2a+3b$, using the method of detached coefficients.

2. Express $\left(1-\dfrac{a^2+b^2-c^2}{2ab}\right)^2$ as a fraction with a numerator of four factors.

3. Solve the equation $\dfrac{4x-11}{x-3}-\dfrac{2x-17}{x-9}=\dfrac{3x-22}{x-7}-\dfrac{x-10}{x-9}$.

4. With the same axes draw the graphs of $y=x+4$ and $y=x^2$. Hence solve the equation $x^2-x-4=0$ as accurately as you can.

5. Two cyclists, riding 9 and 10 miles an hour respectively, start from two places 55 miles apart at noon towards one another. Find graphically, as accurately as you can, their time of meeting, and the times when they are 20 miles apart. Verify your results by algebra.

6. Solve the equations $(x+2y)^2+(2x-y)^2=85$, $xy=4$.

7. From two towns 445 miles apart, two cyclists start on Monday morning to meet each other. One travels at the rate of 48, the other at the rate of 57 miles a day. Find on what day they will meet.

XXX. l.

1. Multiply $2x^3-3x^2+4x-5$ by $3x^2+4x+5$.

2. Prove the identity $\frac{a}{ax-x^2}+\frac{b}{bx-x^2}+\frac{c}{cx-x^2}=\frac{1}{a-x}+\frac{1}{b-x}+\frac{1}{c-x}+\frac{3}{x}$

3. Solve the equations $\frac{2}{x-3}+\frac{1}{y-2}=2$, $\frac{4}{x-3}+\frac{1}{y-2}=3$.

4. Solve the equations $x+y=7$, $xy=4$ by a geometrical method, as accurately as you can.

5. A cycles along a road starting at 15 miles an hour, but diminishing his pace by 3 m. an hour at the end of each hour. B starts at the same time, in the same direction, at 9 m. an hour, increasing his pace by one mile an hour at the end of each hour. Draw in one diagram a graph to give their positions at the end of each hour. Determine when and where they meet again, and how far apart they are in 5 hours.

6. Solve the equations
$$x^2-xy+y^2=21,$$
$$x^2-y^2=9.$$

7. A and B, who live p miles apart, start at the same time to visit each other. If A travel at the rate of q miles in an hour, and B at the rate of r miles in an hour, express in terms of p, q, and r the time which will elapse before they meet.

XXX. m.

1. Multiply $\frac{a^2-ab+b^2}{a^3-3ab(a-b)-b^3}$ by $\frac{a^2-b^2}{a^3+b^3}$.

2. Solve the equation $\frac{5x^2+x-3}{5x-4}=\frac{7x^2-3x-9}{7x-10}$.

3. Find the square root of $x^2+\frac{4a(x^2-3x+a)}{x^2-6x+9}$.

4. A man spends £75 in 64 days. Draw a graph to give his expenditure in any number of days. Write down his expenditure in 17, 35, and 49 days, to the nearest shilling.

5. Draw the graphs of $x^2+y^2-4x-8y=0$ and $2y-x=6$, in the same diagram, and hence solve the equations.

6. Solve the equations
$$(3x+y)^2-(3y+x)^2=24,$$
$$x^2+y^2=5.$$

7. A rectangular grass plot, 8 ft. longer than it is broad, is surrounded by a path 2 ft. 6 in. wide. The cost of making the path, at 1*s*. 6*d*. a square yard, is £3. 2*s*. 6*d*. Find the length and breadth of the plot of grass.

XXX. n.

1. Simplify $\dfrac{a^3-3a^2b+3ab^2-b^3}{a^3-a^2b-ab^2+b^3}$.

2. Solve the equation $\dfrac{(1+x)^3}{1+x^3}=\dfrac{25}{13}$.

3. Resolve into factors (i) $(a^4-b^4)-(a+b)^2(a-b)^2+2b(a^3+b^3)$.
(ii) $x^3-10x^2+31x-30$.

4. Draw the graphs of $y=2x-x^2$, $2x+y=0$, and hence solve the equations.

5. Determine graphically the maximum value of $3-4x^2-12x$. Write down the value of x in that case, and verify your results by algebra.

6. Solve the equations $4x^2-6xy+y^2=11$,
$3y^2-2xy=14$.

7. A walks over a certain course and back again; B starting at the same time walks at half the pace of A over five eighths of the course and back again. A passes B half a mile from the winning post: find the length of the course.

Solve the problem graphically or by algebra.

XXX. p.

1. Divide $ab(x^2+y^2)+(a^2+b^2)xy+(a-b)(x-y)-1$ by $ax+by-1$.

2. Solve the equation $6(x+4)^2+(x-4)^2=5(x^2-16)$.

3. Factorize (i) $a(a+b-c)(a-b+c)-b(b+c-a)(a+b-c)$.
(ii) $x^4-3x^2y^2+y^4$.

4. Draw the graph of $y=x^2-3x$, using a large x unit. Hence solve, as accurately as you can, the equation $x^2-3x=7$.

5. A, starting at noon, cycles 15 miles in the first hour, and diminishes his speed by 2 miles an hour at the end of each hour. B, starting at 2.30 p.m. in his motor car, catches him up at 4.30 p.m. How fast does B travel? Solve the problem graphically.

6. Solve the equations $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=13$,
$\dfrac{1}{y}-\dfrac{1}{x}=1$,
$\dfrac{1}{xy}-\dfrac{2}{z}=0$.

7. A woman has a fifth more apples than pears, but obtains a pound less for her apples when they sell at sixteen a shilling than for her pears, each of which is worth two apples. How many of each kind of fruit has she?

CHAPTER XXXI.

LITERAL EQUATIONS.

169. Instead of numerical coefficients, we sometimes have to deal with coefficients denoted by symbols whose values are supposed to be known. Such coefficients are called literal.

The methods of solution are the same as in dealing with numerical coefficients.

Simple Equations. (One unknown.)

Example 1. Solve the equation

$$\frac{x-a}{a-b}-\frac{x+a}{a+b}=\frac{2ax}{a^2-b^2}.$$

Multiplying both sides by a^2-b^2,

$$(x-a)(a+b)-(x+a)(a-b)=2ax.$$

Removing brackets, and transposing,

$$x(a+b-a+b-2a)=a^2-ab+a^2+ab,$$

$$2x(b-a)=2a^2.$$

Dividing both sides by $2(b-a)$,

$$x=\frac{a^2}{b-a}.$$

Examples. XXXI. a.

Solve the equations:

1. $\dfrac{x+a}{x-b}=1-\dfrac{x}{x-b}.$

2. $\dfrac{a}{bx}-\dfrac{b}{ax}=a^2-b^2.$

3. $\dfrac{a-b}{x-c}=\dfrac{a+b}{x+c}.$

4. $\dfrac{x}{a-2b}=2+\dfrac{x}{2a-b}.$

5. $\dfrac{acx}{b}+\dfrac{abx}{c}-\dfrac{1}{abc}=\dfrac{1}{abc}(1-b^2c^2x).$

6. $\dfrac{x+a}{x-c}+\dfrac{x+c}{x-a}=2.$

7. $x-\dfrac{ax}{a+b}+a=\dfrac{a^2}{a-b}-\dfrac{b^2x}{a^2-b^2}.$

8. $\dfrac{x}{a+c}=\dfrac{x+1}{a+b+c}.$

9. $(x-a-b)^2=x^2-(a-b)^2.$

10. $\dfrac{3x}{a}+2b(a-c)+\dfrac{x}{b}=c(a+b)+\dfrac{2x}{c}.$

11. $\dfrac{1}{x-a}+\dfrac{1}{x-b}=\dfrac{2}{x}.$

12. $\dfrac{q-r}{x-p}+\dfrac{r-p}{x}+\dfrac{p-q}{x-r}=0.$

13. $\dfrac{x-2a}{x+2a}=\dfrac{x-a}{x+a}.$

14. $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-(a+b)}.$

15. $\dfrac{3\{ab-x(a+b)\}}{a+b}+\dfrac{(2a+b)b^2x}{a(a+b)^2}=\dfrac{bx}{a}-\dfrac{a^2b^2}{(a+b)^2}.$

Solve the equations:

*16. $\dfrac{(a^2-1)(ax+1)}{a^2(x+a)}+\dfrac{(a^2+1)(x-a)}{ax+1}=\dfrac{ax+1}{x+a}+\dfrac{a(ax-1)}{ax+1}.$

17. $\dfrac{x}{ax+b}+\dfrac{x}{a+bx}=\dfrac{a+b}{ab}.$ 18. $\dfrac{x-a}{x-b}+\dfrac{x-c}{x-d}=2.$ 19. $\dfrac{x+2a}{x-2b}=\left(\dfrac{x+a}{x-b}\right)^2.$

20. $\dfrac{a}{x+a}+\dfrac{b}{x+b}=\dfrac{a+b}{x+a+b}.$ 21. $\dfrac{x-2b}{a+b}+\dfrac{x-b}{a+2b}=\dfrac{2(x-a)}{3b}.$

22. $(x+a)(x+b)+(x+b)(x+c)=(x+c)(x+d)+(x+d)(x+a).$

23. $\dfrac{1}{x-a}-\dfrac{1}{x-a+c}=\dfrac{1}{x-b-c}-\dfrac{1}{x-b}.$

24. $\dfrac{1}{x-a}-\dfrac{1}{x-b}=\dfrac{a-b}{x^2-ab}.$ 25 $\dfrac{ax}{x-b}+\dfrac{bx}{x-a}=a+b.$

Simple Simultaneous Equations.

***170. Example 1.** Solve the equations $ax+by=p$ .... (1)

$bx-ay=q$ .... (2)

Multiplying (1) by a and (2) by b,

$$a^2x+aby=ap,$$
$$b^2x-aby=bq.$$

Adding, $$x(a^2+b^2)=ap+bq,$$
$$x=\frac{ap+bq}{a^2+b^2}.$$

Instead of substituting for x to find the value of y, it will be simpler to eliminate x from the given equations.

Multiplying (1) by b and (2) by a,

$$abx+b^2y=bp,$$
$$abx-a^2y=aq.$$

Subtracting, $$y(a^2+b^2)=bp-aq,$$
$$y=\frac{bp-aq}{a^2+b^2};$$

$\therefore\ x=\dfrac{ap+bq}{a^2+b^2},\ y=\dfrac{bp-aq}{a^2+b^2}$ is the reqd. solution.

Example 2. Solve the equations

$\dfrac{x}{a}+\dfrac{y}{b}=1,$ .... (1)

$\dfrac{x}{b}+\dfrac{y}{a}=1.$ .... (2)

Subtracting, $$x\left(\frac{1}{a}-\frac{1}{b}\right)+y\left(\frac{1}{b}-\frac{1}{a}\right)=0,$$

i.e. $$x\left(\frac{1}{a}-\frac{1}{b}\right)-y\left(\frac{1}{a}-\frac{1}{b}\right)=0;$$

$$\therefore\ x=y.$$

Substituting in (1) or (2),

$$x\left(\frac{1}{a}+\frac{1}{b}\right)=1,$$

$$x=\frac{ab}{a+b}=y.$$

*Examples. XXXI. b.

Solve the equations :

1. $3(x-a)-2(y+a)=5-4a,$
   $2(x+a)+3(y-a)=4a-1.$

2. $(a+b)x+cy=bc, \quad (b+c)y+ax=-ab.$

3. $ax+by=3(a^2+b^2), \quad x+4b=y+2a.$

4. $ax+by=s, \quad ax-by=t.$

5. $ax-by=a^2, \quad bx-ay=b^2.$

6. $ax+by=a^2+2ab-b^2, \quad bx+ay=a^2+b^2.$

7. $(a+b)x+(c+d)y=bc-ad, \quad (a-b)x+(c-d)y=ad-bc.$

8. $\dfrac{x}{b-c}+\dfrac{y}{c-a}=\dfrac{1}{a-b}, \quad \dfrac{x}{c-a}+\dfrac{y}{a-b}=\dfrac{1}{b-c}.$

9. $a(x+y)-b(x-y)=2a, \quad (a^2-b^2)(x-y)=4ab$

10. $ax-by=2ab, \quad 2bx+2ay=3b^2-a^2.$

11. $x(b-c)+by-c=0, \quad y(c-a)-ax+c=0.$

12. $axy=c(bx+ay), \quad bxy=c(ax-by).$

13. $c^2x+2a^2y=(c+a)(cx+2ay)=(c-a)^2.$

14. $axy+b=(a+c)y, \quad bxy+a=(b+c)y.$

15. $\dfrac{x}{a+b}+\dfrac{y}{a-b}=\dfrac{a^2+b^2}{a^2-b^2}, \quad \dfrac{x}{b}+\dfrac{y}{a}=\dfrac{a^2+b^2}{ab}.$

16. $\dfrac{a}{x}+\dfrac{b}{y}=p, \quad \dfrac{b}{x}+\dfrac{a}{y}=q.$

17. $(a-b)x+(a+b)y=2(a^2-b^2), \quad ax-by=a^2+b^2$

18. $ax+y=c, \quad x+by=d.$

19. $ab(bx-ay)=c(a-b)(a^2+ab+b^2)=c(a^2x-b^2y)$

20. $\dfrac{2x-y}{10a+3b}=\dfrac{x-3y}{4b}=\dfrac{y+b}{2a}.$

21. $(a^2-1)x-2ay=a, \quad 2ax+(a^2-1)y=1.$

22. $by+cz=a, \quad cz+ax=b, \quad ax+by=c.$

23. $\dfrac{x}{a}=\dfrac{y}{b}=\dfrac{z}{c}, \quad lx^2+my^2+nz^2=1.$

24. $a(y+z)=yz, \quad b(z+x)=xz, \quad c(x+y)=xy.$

QUADRATIC EQUATIONS.

171. When the equation has been simplified, the factors can generally be seen by inspection.

Example 1. Solve the equation $x^2-3ax-18a^2=0.$

Factorizing, $$(x-6a)(x+3a)=0;$$

$$\therefore x=6a \text{ or } -3a.$$

Example 2. Solve the equation $ax(x-1)+b(x+1)=2b$.

Removing brackets and re-arranging,

$$ax^2+x(b-a)-b=0.$$

Factorizing,

$$(ax+b)(x-1)=0;$$

$$\therefore \ ax+b=0 \text{ or } x-1=0,$$

$$x=-\frac{b}{a} \text{ or } 1.$$

Examples. XXXI. c.

Solve the equations:

1. $x^2-2ax=15a^2$.

2. $x(5a-x)=6a^2$.

3. $bx\left(a-\frac{1}{x}\right)-c\left(a-\frac{1}{x}\right)=0$.

4. $x^2-(a+b)x+ab=0$.

5. $x^2-2ax+a^2=\frac{1}{a^2}$.

6. $px\left(x-\frac{1}{a}\right)+q\left(x-\frac{1}{a}\right)=0$.

7. $\frac{p-x}{p-a}=\frac{p+a}{p+x}$.

8. $\frac{a^2x^2}{b^2}+1=\frac{2ax}{b}$.

9. $abx^2+1=(a+b)x$.

10. $\frac{abx^2-1}{a-b}=x$.

11. $ax(x-3b)+2(x+2b)ab=16ab^2$.

12. $\frac{a^2x^2}{f^2}-\frac{2ax}{g}+\frac{f^2}{g^2}=0$.

13. $\frac{1}{2}(x+a)^2-\frac{1}{3}(2x-a)^2=\frac{19a^2}{24}$.

14. $\frac{1}{2x-5a}+\frac{5}{2x-a}=\frac{2}{a}$.

15. $x^2-2bx=4a^2+4ab$.

16. $4ax+b^2=4x^2+a^2$.

17. $(a^2-b^2)(x^2+1)=2(a^2+b^2)x$.

18. $\frac{a^2(x-b)}{a-b}+\frac{b^2(x-a)}{b-a}=x^2$.

19. $\frac{\text{[illegible]}}{x+a-1}+\frac{\text{[illegible]}}{x-a+1}=\frac{\text{[illegible]}}{x-1}$.

20. $\frac{1}{x-a}+\frac{1}{x-b}+\frac{1}{x-c}=0$.

21. $4x^2-4ax+a^2=\frac{1}{b^2}$.

22. $\frac{b}{x-a}+\frac{a}{x-b}-2=0$.

23. $\frac{b-x}{a-x}+\frac{a-x}{b-x}=\frac{a}{b}+\frac{b}{a}$.

24. $\frac{ax^2-b}{ax+b}+\frac{a+bx^2}{a-bx}=\frac{2(a^2+b^2)}{a^2-b^2}$.

25. $bx^2+ay^2=a^3+b^3$, $x+y=a+b$.

EQUATIONS IN AN IRRATIONAL FORM.

172. The square root of any quantity may always be regarded as having two values equal in magnitude but of opposite signs. For example, the square root of 49 is ± 7. When, however, such an expression as $\sqrt{2x+3}$ occurs in an equation it is usual to regard it as meaning the *positive* value of the square root of $2x+3$. It might be contended that $\sqrt{4x+7}-\sqrt{4x+3}=2$

was the same equation as $\sqrt{4x+7}+\sqrt{4x+3}=2$; but they are commonly regarded as being different, and instructions are given that after solving an equation of this sort, the answers obtained should be substituted in the original equation to see whether they satisfy it.

Example 1. Solve the equation $\sqrt{4x+7}+\sqrt{4x+3}=6$.

By transposition, $\sqrt{4x+3}=6-\sqrt{4x+7}$. .... (1)

Square; $\therefore\ 4x+3=36-12\sqrt{4x+7}+4x+7$; ....(2)

$$\therefore\ 12\sqrt{4x+7}=36+7-3=40;$$

$$\therefore\ \sqrt{4x+7}=\tfrac{10}{3}.$$

Square; $\therefore\ 4x+7=\tfrac{100}{9}$;

$$\therefore\ 4x=\tfrac{37}{9};\quad \therefore\ x=\tfrac{37}{36}.$$

This root will be found on substitution to satisfy the equation

$$\sqrt{4x+7}+\sqrt{4x+3}=6.$$

Example 2. Solve the equation $\sqrt{2x+3}+\sqrt{x-10}=6$. .... (1)

By transposing, $\sqrt{2x+3}=6-\sqrt{x-10}$.

Squaring, $2x+3=36-12\sqrt{x-10}+x-10$;

$\therefore\ x-23=-12\sqrt{x-10}$. ....(2)

Squaring, $x^2-46x+529=144(x-10)$

$$=144x-1440;$$

$$\therefore\ x^2-190x=-1969;$$

$$\therefore\ x=11 \text{ or } 179.$$

The result 11 satisfies the equation; 179 does not. The fact is that in solving equation (1) we have introduced an additional root through squaring. As we squared equation (2) it would have made no difference if we had written it $x-23=12\sqrt{x-10}$. Thus, in solving (1) we are also solving the equation $\sqrt{2x+3}-\sqrt{x-10}=6$; and this is the equation which is satisfied by the result 179.*

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* This may be expressed in general terms.

If we solve an equation $P=Q$ by squaring, we introduce generally an additional root.

The equation becomes $P^2=Q^2$,

$$\text{i.e. } P^2-Q^2=0,$$

$$\text{i.e. } (P+Q)(P-Q)=0.$$

Thus we have not only the original equation $P-Q=0$, but another one also, viz. $P+Q=0$, i.e. $P=-Q$.

Example 3. Solve $x^2 - x + 5\sqrt{2x^2 - 5x + 6} = \frac{1}{2}(3x + 33)$.

$$2x^2 - 2x + 10\sqrt{2x^2 - 5x + 6} = 3x + 33\,;$$

$$\therefore\ 2x^2 - 5x + 10\sqrt{2x^2 - 5x + 6} = 33.$$

Let $\sqrt{2x^2 - 5x + 6} = y$, *i.e.* $2x^2 - 5x + 6 = y^2$.

Then the equation becomes

$$y^2 - 6 + 10y = 33\,;$$

$$\therefore\ y^2 + 10y - 39 = 0\,;$$

$$\therefore\ (y - 3)(y + 13) = 0,$$

$$i.e.\ \sqrt{2x^2 - 5x + 6} = 3 \text{ or } -13\,;$$

$$\therefore\ 2x^2 - 5x + 6 = 9\,;$$

$$\therefore\ 2x^2 - 5x - 3 = 0.$$

By substitution it will be seen that the negative value (-13) of y will not satisfy the equation.

Thus the question has been reduced to the solution of a quadratic equation.

The following plan is sometimes useful.

Example 4. Solve $\sqrt{2x^2 + 9x - 1} + \sqrt{2x^2 - 7x + 7} = 6.$ .... (1)

Now evidently $2x^2 + 9x - 1 - (2x^2 - 7x + 7) = 16x - 8\,;$ .... (2)

$\therefore$ from (1) and (2) by division we obtain

$$\sqrt{2x^2 + 9x - 1} - \sqrt{2x^2 - 7x + 7} = \frac{8x - 4}{3}\,; \quad \ldots\ldots\ldots\ldots\ldots \quad (3)$$

$\therefore$ by adding (1) and (3)

$$2\sqrt{2x^2 + 9x - 1} = \frac{8x - 4}{3} + 6 = \frac{8x + 14}{3}\,;$$

$$\therefore\ 6\sqrt{2x^2 + 9x - 1} = 8x + 14\,;$$

$$\therefore\ 3\sqrt{2x^2 + 9x - 1} = 4x + 7\,;$$

$\therefore$ by squaring, $18x^2 + 81x - 9 = 16x^2 + 56x + 49\,;$

$$\therefore\ 2x^2 + 25x - 58 = 0\,;$$

$$\therefore\ (2x + 29)(x - 2) = 0\,;$$

$$\therefore\ x = 2 \text{ or } -\frac{29}{2}.$$

Test, as before, to see whether the roots satisfy the equation.

Examples. XXXI. d.

Solve the following equations and verify the solutions by substitution:

1. $\sqrt{2x + 3} = 5.$
2. $\sqrt{3x - 5} = 1.$
3. $\sqrt[3]{4x - 1} = 3.$
4. $5\sqrt{x - 1} = \sqrt{x + 1}.$
5. $\sqrt{x - 1} = \sqrt{x} - 1.$
6. $\sqrt{x^2 - 9} = 4.$
7. $\sqrt{3x^2 - 4x + 9} = 3.$
8. $\sqrt{2x + 3} + \sqrt{2x - 2} = 5.$
9. $\sqrt{7x + 1} - \sqrt{2x} = \sqrt{5x}.$
10. $\sqrt{5x + 9} - \sqrt{3x + 1} = \sqrt{2(x - 6)}.$
11. $\sqrt{2x + 10} + 2\sqrt{x + 6} = 2.$
12. $\sqrt{2x + 8} + 2\sqrt{x + 5} = 2.$

13. $x+5=\sqrt{x+5}+6$.

14. $\sqrt{x+1}+\sqrt{x+8}=7$.

15. $\sqrt{x}-\sqrt{x-(a-b)^2}=a+b$.

16. $x^2=21+\sqrt{x^2-9}$.

17. $\sqrt{ax+b^2}+\sqrt{ax-2ab}=2a+b$.

18. $\sqrt{1+9x}+\sqrt{4x+1}=\sqrt{x+1}$.

19. $\dfrac{5}{\sqrt{x+2}}=\sqrt{x+2}+\sqrt{x-1}$.

20. $\sqrt{5ax+4b}+\sqrt{5ax-4b}=4\sqrt{b}$.

21. $\sqrt{x}+\sqrt{x-7}=\dfrac{21}{\sqrt{x-7}}$.

22. $\sqrt{x+1}+\sqrt{x+4}=\sqrt{x+9}$.

23. $\sqrt{x+2}+\sqrt{x}=\dfrac{4}{\sqrt{x+2}}$.

24. $\sqrt{x+a\sqrt{4x+2a^2}}=a+\sqrt{x}$.

25. $\sqrt{x-a^2}-\sqrt{x-b^2}=b-a$.

26. $x^2+\sqrt{x^2+3x+5}=7-3x$.

27. $x^2+\sqrt{x^2-5x+1}=5x+1$.

28. $x^2+2x+6\sqrt{x^2+2x+5}=11$.

29. $x^2+2x+4\sqrt{x^2+2x+8}=24$.

30. $3x^2-2\sqrt{3x^2-2x+1}=2(x+1)$.

31. $9x-3x^2+4\sqrt{x^2-3x+5}=11$.

32. $2x^2-\sqrt{(x-3)(2x-7)}=13x+9$.

33. $\sqrt{x^2+3x+6}-\sqrt{x^2+3x-1}=1$.

***173.** We now give some miscellaneous equations, of which the following are types.

Example 1. Solve the equations:

$$x+y+z=19, \quad\quad (1)$$
$$x^2+y^2+z^2=133, \quad\quad (2)$$
$$yz=x^2. \quad\quad (3)$$

Squaring (1), subtracting (2) from it, and dividing by 2,

$$xy+yz+zx=114, \quad\quad (4)$$

$\therefore$ from (3) $\qquad x(y+x+z)=114,$

and from (1) $\qquad x=6.$

Substituting this value of x and solving for y and z we obtain the following solutions

$$x=6,\ 6,$$
$$y=9,\ 4,$$
$$z=4,\ 9.$$

Example 2. Solve the equations:

$$x(y+z)=7, \quad\quad (1)$$
$$y(x+z)=4, \quad\quad (2)$$
$$z(x+y)=5. \quad\quad (3)$$

Adding (1), (2) and (3), and dividing by 2,

$$xy+yz+zx=8.$$

Subtracting (1), (2) and (3) from this, in succession,

$$yz=1,$$
$$xz=4,$$
$$xy=3.$$

Whence by multiplication, $\qquad x^2y^2z^2=12$

$$xyz=\pm\sqrt{12}$$

$$\therefore\ x=\pm 2\sqrt{3},\ y=\frac{\pm\sqrt{12}}{4},\ z=\frac{\pm\sqrt{12}}{3}.$$

Example 3. Solve the equations :

$$x^2+2yz=48, \quad\ldots\ldots (1)$$
$$y^2+2zx=48, \quad\ldots\ldots (2)$$
$$z^2+2xy=48. \quad\ldots\ldots (3)$$

Adding and taking the square root of both sides,

$$x+y+z=\pm 12. \quad\ldots\ldots (4)$$

Subtracting (2) from (1) and factorizing,

$$(x-y)(x+y-2z)=0.$$
$$\therefore\ x=y \text{ or } x+y=2z.$$

(i) If $x=y$, from (1) $\quad x^2+2xz=48,$

and from (3) $\quad z^2+2x^2=48,$

whence $\quad z=x.$

$$\therefore\ x=y=z, \text{ and from (1) or (2) or (3)}$$
$$x=\pm 4=y=z.$$

(ii) If $\quad x+y=2z,$

from (4) $\quad z=\pm 4=x=y$ as before;

$$\therefore\ x=y=z=\pm 4 \text{ are the only solutions.}$$

*Examples. XXXI. e.

Solve the equations :

1. $(x+y)^3+z^3=1125,$
$x+y+z=15,$
$xy=24.$

2. $xz=y^2,$
$x+y+z=13,$
$x^2+y^2+z^2=91.$

3. $x^3-2x=\frac{7}{8}.$

4. $\dfrac{x+y}{x-y}+10\dfrac{x-y}{x+y}=7,$
$xy^1=3.$

5. $xy+\dfrac{x}{y}=10,$
$xy^2-x=6y.$

6. $x+y=a+b,$
$\dfrac{a}{x}+\dfrac{b}{y}=2.$

7. $x^2+xy+y^2-2x^2+3xy+y^2=c^2.$

8. $x+y+z=7,$
$xy+xz+yz=14,$
$xyz=8.$

9. $\dfrac{x+a}{x+b}+\dfrac{x-a}{x-b}=\dfrac{a}{b}.$

10. $\dfrac{(x-a)(x-b)}{(x-c)(x-d)}=\dfrac{x-a-b}{x-c-d}.$

11. $(ax+by)^2+(ay-bx)^2=2\left(\dfrac{a}{b}+\dfrac{b}{a}\right)^2,$
$\dfrac{x}{y}+\dfrac{y}{x}=2\dfrac{a^2+b^2}{a^2-b^2}.$

12. $x(y+z)=5,$
$y(x+z)=4,$
$z(x+y)=3.$

13. $(x+y)(x+z)=1,$
$(y+z)(y+x)=4,$
$(z+x)(z+y)=9.$

14. Find the rational solutions of the equations,

$$\frac{x^2}{y^2}+\frac{y^2}{x^2}+\frac{x}{y}-\frac{y}{x}=\frac{106}{9},\ xy=3.$$

INDETERMINATE EQUATIONS.

***174.** When we have but one equation involving *two* variables we can generally find any number of solutions. (Art. 57.)

Such equations, however, often admit of only a limited number of *positive integral* solutions.

Example. *Find the positive integral solutions of the equation*

$$5x + 12y = 193. \quad \text{.....................}(1)$$

By putting x or $y = 0$, 1, 2 and so on, one pair of roots can generally be found without difficulty.

Here we see *by trial* that one pair of roots is given by $x=5$, $y=14$.

$$i.e. \quad 5 \times 5 + 12 \times 14 = 193. \quad \text{.....................}(2)$$

Subtracting (2) from (1), $5(x-5) + 12(y-14) = 0$.

$$5(x-5) = 12(14-y),$$

$$\frac{x-5}{14-y} = \frac{12}{5}.$$

Now $\frac{12}{5}$ is in its lowest terms, and x and y must be positive integers,

$$\therefore \ x - 5 = 12p,$$

and $14 - y = 5p$, where p is an integer.

$$i.e. \quad x = 5 + 12p, \quad \text{.....................}(3)$$

$$y = 14 - 5p. \quad \text{.....................}(4)$$

From (3) p cannot be <0, for then x would be negative.

...... (4) .... >2 .... y ....

$\therefore$ 0, 1, 2 are the only admissible values of p

Hence from (3) and (4) the only positive integral solutions of the given equations are

$$(p=0) \ \left.\begin{array}{l} x=5 \\ y=14 \end{array}\right\} \quad (p=1) \ \left.\begin{array}{l} x=17 \\ y=9 \end{array}\right\} \quad (p=2) \ \left.\begin{array}{l} x=29 \\ y=4 \end{array}\right\}.$$

***175.** *In how many ways can a bill of £2. 7s. 6d. be paid in half-crowns and half-sovereigns?*

Let x be the number of half-crowns and y the number of half-sovereigns required to pay the bill.

Then
$$\tfrac{5}{2}x + 10y = 47\tfrac{1}{2}$$
$$5x + 20y = 95$$
$$x + 4y = 19. \qquad\qquad \text{(1)}$$

Now x and y must evidently be positive integers, and we see that the equation is satisfied by $x = 3$, $y = 4$,

$$\text{i.e. } 3 + 4 \times 4 = 19. \qquad\qquad \text{(2)}$$

Subtracting (2) from (1)

$$x - 3 + 4(y - 4) = 0,$$
$$x - 3 = 4(4 - y),$$
$$\frac{x-3}{4-y} = \frac{4}{1};$$
$$\therefore\ x - 3 = 4p,$$
$$4 - y = p, \text{ where } p \text{ is an integer.}$$
$$\text{i.e. } x = 3 + 4p, \qquad\qquad \text{(3)}$$
$$y = 4 - p. \qquad\qquad \text{(4)}$$

From (3) p cannot be < 0, for then x would be negative.

... (4) .... > 4 .... y ....

$\therefore$ 0, 1, 2, 3, 4 are the only admissible values of p;

i.e. there are *five* ways of paying the bill.

This includes as a solution the case when no half-sovereigns are used, for when $p = 4$, $y = 0$.

GRAPHICAL SOLUTION OF INDETERMINATE EQUATIONS.

***176. Example.** Find the positive integral solutions of the equation

$$3x+2y=30.$$

Use a half-inch unit.

When $\qquad x=0,\ y=15,$

$\qquad\qquad y=0,\ x=10.$

Joining the points (0, 15), (10, 0) by a str. line, we have the graph of the equation $3x+2y=30$.

The only points, whose co-ordinates are positive integers, through which the line passes, will be seen to be the points

(8, 3), (6, 6), (4, 9), (2, 12), not counting zero values.

$\therefore$ these are the reqd. solutions

*Examples. XXXI. f.

Find the positive integral solutions of:

1. $2x+5y=35.$ **2.** $2x+3y=15.$ **3.** $5x+2y=27.$

4. $7x+3y=73.$ **5.** $9x+5y=33.$ **6.** $7x+13y=207.$

How many positive integral solutions are there of:

7. $2x+13y=185.$ **8.** $2x+11y=165.$

9. $4x+9y=207.$ **10.** $7x+3y=119$

11. Prove that the equation $7x-5y=16$ has an infinite number of positive integral solutions.

Use graphical methods to find the positive integral solutions of

12. $3x+2y=17.$ **13.** $5x+y=18.$ **14.** $3x+4y=48$

15. $2x+7y=23.$ **16.** $2x+3y=30.$

Find graphically, or by algebra, all integral solutions of the following equations which have positive values of x and negative values of y:

17. $x-2y=12.$ **18.** $2x-3y=24.$ **19.** $x-y=4.$

Find graphically, or by algebra, all integral solutions of the following equations which have negative values of x and y:

20. $2x+3y+24=0.$ **21.** $4x+3y+24=0.$ **22.** $x+2y+12=0.$

23. A man bought a number of books at 5*s*. each, and a number at 7*s*. each, and spent 38*s*.: how many of each did he buy?

24. A man bought a number of geese at 7*s*. each, and a number of turkeys at 11*s*. each, and spent £4. 6*s*.: how many of each did he buy?

25. In how many ways can I pay a bill of 31*s*. in sixpences and shillings, excluding zero solutions?

26. Divide 59 into two parts so that one may be a multiple of 9 and the other of 4.

27. A has only four-shilling pieces, and B only half crowns. What is the simplest way in which A can pay B the sum of 35*s*.?

28. In how many ways can I pay a bill of 37*s*., if I have only florins and half-crowns in my pocket?

29. The sum of two fractions is $2\frac{3}{28}$ and their denominators are 4 and 7. Find all the solutions of the problem.

30. Find general formulae to represent all the integral solutions of the equation $9x - 13y = 63$.

31. A has 25 four-shilling pieces, and B 25 half-crowns: in how many ways can A pay B the sum of 37*s.*?

32. Find the positive integral solution of the equation $5x + 13y = 227$, for which the value of x is largest.

33. A man exchanges a number of geese at 7*s.* each, for a number of turkeys at 13*s.* each, and receives £4. 13*s.* in cash. Find the number of ways in which the exchange can be made, a condition being made that the man shall not take more than 20 turkeys.

CHAPTER XXXII.

THEORY OF QUADRATIC EQUATIONS.

177. *To prove that a quadratic equation cannot have more than two roots.*

If possible, let the general quadratic equation

$$ax^2 + bx + c = 0$$

have *three different* roots α, β, γ.

By hypothesis, each of these values of x satisfies the equation, $\therefore$ by substitution

$$a\alpha^2 + b\alpha + c = 0, \quad\ldots\ldots(1)$$

$$a\beta^2 + b\beta + c = 0, \quad\ldots\ldots(2)$$

$$a\gamma^2 + b\gamma + c = 0. \quad\ldots\ldots(3)$$

Subtracting (2) from (1), $a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0.$

Dividing by $\alpha - \beta$, which by hypothesis is not equal to zero,

$$a(\alpha + \beta) + b = 0. \quad\ldots\ldots(4)$$

In the same way, subtracting (3) from (1) and dividing by $\alpha - \gamma$,

$$a(\alpha + \gamma) + b = 0. \quad\ldots\ldots(5)$$

Subtracting (5) from (4), $a(\beta - \gamma) = 0$;

$$\therefore\ a = 0 \text{ or } \beta - \gamma = 0,$$

which is impossible, for a is not equal to zero, nor is β equal to γ, by hypothesis.

$\therefore$ the quadratic cannot have more than two roots.

178. The square root of a negative quantity cannot be found. It is said to be '*imaginary*,' or '*unreal*,' or '*impossible*.'

The quadratic equation $ax^2+bx+c=0$, will have

(1) *real and different roots if $b^2>4ac$,*

(2) *real and equal roots if $b^2=4ac$,*

(3) *imaginary roots if $b^2<4ac$.*

We have seen (Art. 149), that the solution of this equation may be thus written:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

(1) If $b^2>4ac$, b^2-4ac is positive, and the value of $\sqrt{b^2-4ac}$ may be found;

$\therefore$ we then have two real and different roots,

$$\text{viz. } \frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

(2) If $b^2=4ac$, $b^2-4ac=0$;

$\therefore$ $x=-\dfrac{b}{2a}$ is the only solution;

in other words the roots are equal, and each equal to

$$-\frac{b}{2a}.$$

(3) If $b^2<4ac$, b^2-4ac is negative, and the value of

$\sqrt{b^2-4ac}$ is imaginary.

Hence the equation in that case has no real roots.

By means of the above we can determine the *nature* of the roots of a quadratic, without actually effecting its solution.

The student must be careful to distinguish between *irrational* and *imaginary* roots.

If the roots of $ax^2+bx+c=0$ are rational, b^2-4ac must be a perfect square.

The roots of $x^2-2x-2=0$ are $1+\sqrt{3}$, and $1-\sqrt{3}$.

These are real but *irrational.*

The roots of x^2-2x+4 are $1+\sqrt{-3}$, and $1-\sqrt{-3}$.

These are *imaginary.*

179. *The roots of $ax^2+bx+c=0$ are equal, but of opposite sign if $b=0$.*

The roots are equal but of opposite sign,

if $$\frac{-b+\sqrt{b^2-4ac}}{2a}=-\left[\frac{-b-\sqrt{b^2-4ac}}{2a}\right]$$

$$=\frac{b+\sqrt{b^2-4ac}}{2a};$$

i.e. if $$\frac{2b}{2a}=0;$$

i.e. if $$b=0.$$

Example 1. When we solve the equation $x^2+px-q^2=0$, the expression under the radical sign

$$=p^2+4q^2, \qquad (b^2-4ac)$$

which is positive.

$\therefore$ the roots of the equation are real and different for all values of p and q.

Example 2. When we solve the equation $5x^2-2x+4=0$, the quantity under the radical sign

$$=4-4\times 20, \text{ which is negative.}$$

$\therefore$ the equation has imaginary roots.

If we drew the graph of $y=5x^2-2x+4$, as in Art. 151, we should find that the curve does not meet the axis of x, *i.e.* no real value of x can be found which will make $5x^2-2x+4$ vanish.

Example 3. When we solve the equation $2x^2-px+8=0$, the expression under the radical sign

$$=p^2-4\times 16=p^2-64.$$

$$\therefore \text{ if } p^2=64, \textit{ i.e. } \text{if } p=\pm 8,$$

the roots of $2x^2-px+8=0$ are equal.

180. *In the quadratic equation $ax^2+bx+c=0$,*

(1) *the sum of the roots* $=-\dfrac{b}{a}$,

(2) *the product of the roots* $=\dfrac{c}{a}$.

Let α and β be the roots.

$$\alpha=\frac{-b+\sqrt{b^2-4ac}}{2a},$$

$$\beta=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Adding, $\alpha+\beta=-\dfrac{b}{a}.$

Multiplying, $\alpha\beta=\dfrac{b^2-(b^2-4ac)}{4a^2}$ $[(p+q)(p-q)=p^2-q^2]$

$$=\frac{4ac}{4a^2}$$

$$=\frac{c}{a}.$$

If we write the equation in the form $x^2+\dfrac{bx}{a}+\dfrac{c}{a}=0$, we may express these results as follows:

When the coefficient of $\mathbf{x}^2$ in a quadratic equation is unity,

(1) the sum of the roots is equal to the coefficient of x with the sign changed;

(2) the product of the roots is equal to the constant term.

These results are of the greatest importance, and will be found most useful in solving problems concerned with the roots of quadratic equations.

181. *If α and β are the roots of $ax^2+bx+c=0$,*

$$ax^2+bx+c=a(x-\alpha)(x-\beta).$$

$$ax^2+bx+c=a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right)$$

$$=a\left[x^2-(\alpha+\beta)x+\alpha\beta\right]$$

$$=a(x-\alpha)(x-\beta).$$

In the same way, if α and β are the roots of $x^2+px+q=0$,

$$x^2+px+q=(x-\alpha)(x-\beta).$$

Example 1. The quadratic whose roots are -5 and 6 is

$$(x+5)(x-6)=0,$$

or $$x^2-x-30=0.$$

Example 2. If α and β are the roots of $x^2-px+q=0$, find the values of (1) $\alpha-\beta$, (2) $\alpha^2+\beta^2$, (3) $\alpha^3+\beta^3$.

(1) $$\alpha+\beta=p, \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

$$\alpha\beta=q. \quad\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Squaring (1) and subtracting four times (2),

$$(\alpha-\beta)^2=p^2-4q;$$

$$\therefore \alpha-\beta=\pm\sqrt{p^2-4q}.$$

(2) Squaring (1) and subtracting twice (2),

$$\alpha^2+\beta^2=p^2-2q.$$

(3) Squaring (1) and subtracting three times (2),

$$\alpha^2-\alpha\beta+\beta^2=p^2-3q.$$

Multiplying this with (1),

$$\alpha^3+\beta^3=p(p^2-3q).$$

Example 3. If α and β are the roots of $ax^2+bx+c=0$, form the equation whose roots are $\frac{1}{\alpha}$, and $\frac{1}{\beta}$.

The sum of the roots of the reqd. equation $=\frac{1}{\alpha}+\frac{1}{\beta}$

$$=\frac{\alpha+\beta}{\alpha\beta}=-\frac{b}{a}\div\frac{c}{a}=-\frac{b}{c}.$$

The product of the roots $=\frac{1}{\alpha\beta}=\frac{a}{c}$.

$\therefore$ the reqd. equation is

$$x^2+\frac{bx}{c}+\frac{a}{c}=0,$$

or

$$cx^2+bx+a=0.$$

182. *If a is positive and α, β are real roots of the equation $ax^2+bx+c=0$, the expression ax^2+bx+c vanishes when $x=\alpha$ or β, and is positive for all other values of x except for those lying between α and β.*

(1) The values α and β satisfy the equation;

$\therefore$ the expression ax^2+bx+c is zero when $x=\alpha$ or β.

(2) $$\alpha+\beta=-\frac{b}{a},\quad \alpha\beta=\frac{c}{a}. \qquad \text{(Art. 180.)}$$

$$\therefore\ ax^2+bx+c=a\left(x^2+\frac{bx}{a}+\frac{c}{a}\right)$$

$$=a[x^2-(\alpha+\beta)x+\alpha\beta]$$

$$=a(x-\alpha)(x-\beta). \qquad \text{.................(1)}$$

Let α be greater than β.

When $x>\alpha$, $x-\alpha$ is positive and $x-\beta$ is positive;

$\therefore$ from (1) ax^2+bx+c is positive.

When $x<\alpha$ but $>\beta$, $x-\alpha$ is negative,

and $x-\beta$ is positive;

$\therefore$ from (1) ax^2+bx+c is negative.

Lastly, when $x<\beta$, $x-\alpha$ is negative,

and $x-\beta$ is negative;

$\therefore$ from (1) ax^2+bx+c is positive.

$\therefore\ ax^2+bx+c=0$, when $x=\alpha$ or β,

is negative when x lies between α and β,

and is positive for all other values of x.

It follows that if a is negative and α and β are the roots of $ax^2+bx+c=0$, the expression ax^2+bx+c is zero when $x=\alpha$ or β, negative for all other values of x except for those lying between α and β.

Example 1. To prove *graphically* that the expression x^2+x-6

(i) vanishes when $x=2$ or -3,

(ii) is negative when x lies between 2 and -3,

(iii) is positive for all other values of x.

(i) If we draw the graph of $y=x^2+x-6$ as in Art. 151, we shall see that the curve cuts the axis of x where $x=2$ and $x=-3$.

(ii) When x lies between these values, y is negative.

(iii) For all other values y is positive.

Example 2. Show that $\dfrac{x^2-3x+4}{x^2+3x+4}$ can never be greater than 7 nor less than $\frac{1}{7}$ for real values of x.

Let $\dfrac{x^2-3x+4}{x^2+3x+4}=u$.

Multiplying up and rearranging as a quadratic for x,

$$x^2(1-u)-3x(1+u)+4(1-u)=0.$$

When we solve this quadratic for x, the expression under the radical sign

$$\begin{aligned} &=9(1+u)^2-16(1-u)^2 \qquad (b^2-4ac)\\ &=-7+50u-7u^2\\ &=(-7+u)(1-7u)\\ &=7(u-7)(\tfrac{1}{7}-u). \end{aligned}$$

Hence if $u>7$, $u-7$ is positive, and $\frac{1}{7}-u$ is negative.

$\therefore$ the expression under the radical sign is negative and x is imaginary.

If $u<7$ but $>\frac{1}{7}$, $u-7$ is negative and $\frac{1}{7}-u$ is negative.

$\therefore$ the expression under the radical sign is positive, and x is real.

If $u<\frac{1}{7}$, $u-7$ is negative, and $\frac{1}{7}-u$ is positive.

$\therefore$ the expression under the radical sign is negative and x is imaginary.

Thus for real values of x, u cannot be greater than 7 or less than $\frac{1}{7}$.

183. *Find the condition that the equations $ax^2+bx+c=0$ and $a'x^2+b'x+c'=0$ may have a common root.*

Let α be a common root of the equations.

Then by substitution $\quad a\alpha^2+b\alpha+c=0, \quad \text{.......................} \quad (1)$

$$a'\alpha^2+b'\alpha+c'=0. \quad \text{.......................} \quad (2)$$

Multiplying (1) by b', and (2) by b, and subtracting,

$$a^2(ab'-a'b)+b'c-bc'=0,$$

or
$$a^2=\frac{bc'-b'c}{ab'-a'b}. \quad \text{.........................(3)}$$

Again multiplying (1) by a', and (2) by a, and subtracting,

$$a(a'b-ab')+a'c-ac'=0,$$

or
$$a=\frac{a'c-ac'}{ab'-a'b}. \quad \text{.........................(4)}$$

$\therefore$ from (3) and (4) $\dfrac{bc'-b'c}{ab'-a'b}=\left(\dfrac{a'c-ac'}{ab'-a'b}\right)^2,$

or. $(ab'-a'b)(bc'-b'c)=(a'c-ac')^2$; the reqd. condition

Examples. XXXII.

Form the equations whose roots are :

1. $2,\ 5$.
2. $4,\ -5$.
3. $\frac{1}{2},\ -\frac{1}{2}$.
4. $0,\ -3$.
5. $2a,\ -3a$.
6. $a+1,\ a-1$.
7. $1+\frac{1}{a},\ 1-\frac{1}{a}$.
8. $m\pm\sqrt{m^2-n}$.
9. $\dfrac{-m\pm\sqrt{m^2-4ln}}{2l}$.
10. $3+\sqrt{3},\ 3-\sqrt{3}$.
11. $\dfrac{4-\sqrt{3}}{5},\ \dfrac{4+\sqrt{3}}{5}$.

12. For what value of k will the roots of $x^2-10x=k$ be equal?

13. What is the condition that the roots of the equation $x^2-px+q=0$ may be rational?

14. Prove that the roots of $x^2-3x+k=0$ will be imaginary if k is greater than $2\frac{1}{4}$.

15. Solve the equation $x^2-px+q=0$, and hence find (1) the sum of the roots, (2) the product of the roots

16. If α and β are the roots of $ax^2+bx+c=0$, find the values of (1) $\alpha-\beta$, (2) $\alpha^2+\beta^2$, (3) $\alpha^3+\beta^3$, (4) $\alpha^4+\beta^4$.

17. If α and β be the roots of the equation $x^2-px+q=0$, form the equation whose roots are 2α, 2β.

If α and β be the roots of the equation $ax^2+bx+c=0$, determine the equation whose roots are :

18. $-\alpha,\ -\beta$.
19. $3\alpha,\ 3\beta$.
20. $\frac{\alpha}{\beta},\ \frac{\beta}{\alpha}$.
21. $\frac{2\alpha}{\beta},\ \frac{2\beta}{\alpha}$.
22. $2\beta-\alpha,\ 2\alpha-\beta$.
23. $\frac{\alpha^2}{\beta},\ \frac{\beta^2}{\alpha}$.

24. Find the numerical value of a in the equation $ax^2+2x+3a=0$, when the sum of its roots is equal to their product.

25. If one root of the equation $ax^2+bx+c=0$, is double the other, prove that $9ac=2b^2$.

26. Form an equation whose roots shall be $\frac{\alpha^2}{\beta}$, $\frac{\beta^2}{\alpha}$, where α, β are the roots of the equation $x^2=px+p^2$.

27. If α, β be the roots of the equation $ax^2+bx-a=0$, determine the equation whose roots are $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$.

28. Find the sum of the cubes of the roots of $x^2+px+q=0$

29. If α, β be the roots of the equation $px^2+qx+r=0$, find the equation whose roots are $\alpha+\beta$, $\alpha\beta$ Find also the value of $\alpha^4+\beta^4$.

30. If α, β be the roots of the equation $ax^2+bx+c=0$, form the equation whose roots are α^2 and β^2

31. Find the quadratic equation whose roots are the squares of the roots of the equation $x^2=px+q$

32. Prove that the equation $x^2-2(k+2)x-k^2=0$, cannot have equal roots for any real value of k For what value of k will the roots be equal but of opposite sign?

33. If α, β be the roots of the equation $x^2+px+q=0$, prove that x^2+px+q will be a negative quantity, if x be put equal to $\frac{1}{3}\alpha+\frac{2}{3}\beta$

34 Find the condition that the two quadratics $x^2+px+q=0$, $x^2+p'x+q'=0$, may have a common root

35 If α, β be the roots of the equation $x^2+px+q=0$, prove that $\alpha^4+\beta^4=(p^2-2q)^2-2q^2$

36. Show that one of the roots of the equation $px^2+qx+r=0$, will be double one of the roots of the equation $rx^2+qx+p=0$, if either $r=2p$ or $2p+r=\pm q\sqrt{2}$

37. If α, β be the roots of the equation $x^2-px+q=0$, prove that $\alpha^5+\beta^5=p^5-5p^3q+5pq^2$.

38. Prove that, if one of the equations

$$x^2-x(3c-b)+bc=0, \quad x^2-x(5c-b)+4c^2=0,$$

has equal roots, so has the other

39. If p, q be the roots of the equation $ax^2+2bx-c=0$, find the equation whose roots are p^3, q^3

40. One root of the equation $x^2+ax+b=0$ is double of the other; and one root of the equation $x^2+ax+c=0$ is equal to three times its other root. Find the value of $\frac{b}{c}$.

41. Prove that the roots of one of the two equations

$$8a^2x(2x-1)+b^2=0, \quad 4a^2x^2+b^2(4x+1)=0,$$

must be imaginary

42. If $ax^2+bx+c=0$, $bcx^2+cax+ab=0$ have a common root, and if $a+b+c=0$, prove that

$$b^4(a-c)^2=a^2c^2(a-b)(b-c)$$

43. The roots of the quadratic $ax^2+bx+c=0$ are x_1, x_2; find in terms of a, b, c, the values of (1) $(ax_1+b)(ax_2+b)$, (2) $(bx_1+c)(bx_2+c)$.

44. If x_1, x_2 be the roots of the equation $ax^2+bx+c=0$, find, in terms of a, b, c, the value of

$$\frac{1}{(b+ax_1)^2}+\frac{1}{(b+ax_2)^2}.$$

45. Prove that, for real values of x, the expression $\dfrac{x^2+3x-15}{x-5}$ can have all numerical values except such as lie between 3 and 23.

46. Prove that $\dfrac{x^2+x+1}{x^2+1}$ cannot be greater than $\frac{3}{2}$, nor less than $\frac{1}{2}$, for real values of x.

47. Prove that $\dfrac{x^2-2x+4}{x^2+2x+4}$ cannot be greater than 3 or less than $\frac{1}{3}$, for real values of x.

48. For real values of x, prove that the expression $\dfrac{4x^2-5x+10}{3(x-2)}$ cannot lie between 9 and $-1\frac{2}{3}$.

49. Find the greatest value which the expression $x+\sqrt{6ax-7a^2-x^2}$ can have for real values of x.

50. Find the minimum value of $\dfrac{x^2-x+1}{x^2+x+1}$, for real values of x.

CHAPTER XXXIII.

Examples. XXXIII. a

1. Resolve into real elementary factors:

(i) $6x^2-23xy+20y^2$. (ii) $x^4-7x^2y^2+y^4$. (iii) x^6-1.

2. Simplify $\dfrac{9}{x^2-x-20}-\dfrac{7}{x^2+x-12}-\dfrac{2}{x^2-8x+15}$.

3. Find the squares of $x+y+2z-1$, and of $x+y-2z-1$. What is the value of the difference of these squares when $z=\frac{1}{2}(x+y)$?

4. Find the L.C.M. of x^5-xy^4, x^9+x^8y, $x^6+y^6+x^2y^2(x^2+y^2)$.

5. Solve the equations (i) $27x^2-57x=14$.

(ii) $x^2+y^2=5$, $x^2-y^2=\dfrac{3xy}{2}$.

6. A travels 42 miles in $5\frac{1}{4}$ hours. Find, graphically, how long he takes to travel 35 miles, and 29 miles. How far did he travel in 2 hrs. 36 min.?

7. Solve the equations

$$x+2y-z+4=0,$$
$$3x+4y+z-1=0,$$
$$5x+6y-3z+18=0.$$

8. If α, β are the roots of the equation $x^2+px+q=0$, form the equation whose roots are $\alpha+2\beta$, $\beta+2\alpha$.

XXXIII. b.

1. Find the factors of (i) $x^2+16x+63$.

(ii) $y^3-43a^2y+42a^3$.

(iii) $x^7-14x^5+49x^3-36x$.

2. Find the square root of $9x^4 - 42x^3 + 37x^2 + 28x + 4$.

3. Simplify $\dfrac{\dfrac{1}{x-a} - \dfrac{1}{x+a} - \dfrac{2a}{x^2+a^2}}{\dfrac{1}{x^3-a^3} - \dfrac{1}{x^3+a^3}}\left(\dfrac{1}{x^2+ax+a^2} + \dfrac{1}{x^2-ax+a^2}\right)$.

4. Solve the equations (i) $\dfrac{a}{bx} + \dfrac{b}{ax} = a^2 + b^2$.

(ii) $(x-10)(x-7) + (2x-9)(x-8) = 103$.

5. A person after paying income-tax of 6*d.* in the £ gave away one-thirteenth part of the remainder, and then had £540 left. What was his original income?

6. On an examination paper of maximum 58 the marks gained by six candidates were 52, 47, 41, 36, 24, 12. Draw a graph to raise the maximum to 100, and read off the raised marks of the candidates. Test one of your results.

7. Employ the Remainder Theorem to prove that $x^4 - 4x^3 + 2x^2 + x + 6$ is exactly divisible by $x^2 - 5x + 6$.

XXXIII. c.

1. Remove the brackets in $7a + 6[b - 5\{c + 4(b - 3(a+2c))\}]$ and find its value when $a=2$, $b=3$, $c=1$.

2. Simplify $\dfrac{1}{x^2-4x+3} - \dfrac{4}{x^2+2x-15} + \dfrac{3}{x^2+4x-5}$.

3. Find the H.C.F. of $x^4 - 8x^3 + 13x^2 - 30x + 8$
and $x^4 - 4x^3 - 11x^2 - 50x + 16$.

4. Solve the equation $\dfrac{\dfrac{2x-1}{3} - \dfrac{4x^2-1}{x+3}}{x+1-\dfrac{5x^2-9}{3(x-3)}} = \dfrac{x-3}{x+3} \cdot \dfrac{10x+1}{2x+3}$.

5. Solve the equations:

(i) $(a+b)(c+x) + (b+c)(a+x) = (c+a)(b+x)$.

(ii) $x + y = 3$, $\dfrac{2}{x} + \dfrac{1}{y} = 2$.

6. I bought a horse and carriage for £80. I sold the horse at a profit of 20 per cent., and the carriage at a loss of 4 per cent., and found that on the whole transaction I had gained 5 per cent. What was the original cost of the horse?

7. Determine the values of k for which the equation

$$12(k+2)x^2 - 12(2k-1)x - 38k - 11 = 0$$

will have equal roots.

XXXIII. d.

1. Divide $x^5 + x^4 + 4x^3 + 21x^2 + 23x - 40$ by $x^2 + 4x + 5$, using the method of detached coefficients.

2. Simplify $\left\{\dfrac{a^3}{b^3} - \dfrac{b^3}{a^3} - 3\left(\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2}\right) + 5\right\} \div \left(\dfrac{a}{b} - 1 - \dfrac{b}{a}\right)^2$.

3. Find the square root of $4x^4 + 12x^3 - 11x^2 - 30x + 25$.

4. A man travels at the rate of x feet per second.
(i) How many yards does he travel per minute?
(ii) ... . miles .... hour?
(iii) .... in y hours?
(iv) How long does he take to travel y miles?

5. Solve the equations:

$$\text{(i)}\ \frac{7x}{1-\frac{2x-12}{3x-5}}=\frac{48}{1-\frac{1}{x}}.$$

$$\text{(ii)}\ \frac{5}{x}-\frac{3}{y}=9,\ 3y+2x=13xy.$$

6. A man on a bicycle, who travels at the rate of 10 miles an hour, and another walking at the rate of 4 miles an hour, start at the same time and from the same point to go round a field a quarter of a mile in circumference in the same direction. Find how soon the bicyclist is one-quarter of the whole circumference ahead of the walker.

7. Trace the graph of $y=3x-x^2$, and deduce the value of x when the expression $3x-x^2$ is a maximum. What is the maximum value of the expression?

XXXIII. e.

1. Show that x^6+a^6 is divisible by $x^2+px+\frac{p^2}{3}$ if $p^6-27a^6=0$.

2. Find the product of $x-y$, $x+y$, x^2-xy+y^2, x^2+xy+y^2.

3. Find the square root of $n(n+1)(n+2)(n+3)+1$.

4. Express $\dfrac{\frac{1}{x}+\frac{1}{y-z}}{\frac{1}{x}-\frac{1}{y-z}}\left\{1-\frac{y^2+z^2-x^2}{2yz}\right\}$ in its simplest form.

5. Employ the Remainder Theorem to prove that

$$1-x^2-2x^3-2x^4-x^5+x^7$$

is exactly divisible by $x+1$ and by x^2+1.

6. Solve the equations

$$\text{(i)}\ \frac{a(a-x)}{b}-\frac{b(b+x)}{a}=x.$$

$$\text{(ii)}\ \frac{3}{3-x}=5-\frac{2}{2-x}\ \text{(correct to two decimal places).}$$

7. Two travellers, one of whom travels 3 miles an hour faster than the other, set out to meet one another, starting simultaneously from two towns which are 216 miles apart. They meet after a lapse of 8 hours. Find the rate at which each of them travels.

8. Divide 1 into two fractions such that the sum of their cubes is $\frac{1}{3}$.

XXXIII. f.

1. Divide $(x+y)^4+(x^2-y^2)^2+(x-y)^4$ by $3x^2+y^2$.

2. Resolve each of the following into three real factors:

$$4x^3-23x^2+28x,\quad y^4+11y^2-180,\quad a^6+27b^6.$$

3. Solve the equations:

(i) $\dfrac{x+a}{x-a}-\dfrac{x-b}{x+b}=\dfrac{2(a+b)}{x}.$

(ii) $x^2+xy=28, \quad xy+y^2=21.$

4. Given that a, β are the roots of $x^2+px+q=0$, find the roots of $x^2+4px+16q=0$.

5. Prove that the difference of the squares of two consecutive numbers is equal to the sum of the numbers.

6. A, walking uniformly but taking a rest of 20 minutes when he has gone half-way, does 5 miles in an hour. B, starting at the same time, and taking no rest, passes A $3\frac{1}{2}$ miles from the start. Find, by the graphical method, how long B takes to walk the $3\frac{1}{2}$ miles.

7. Show, by any method, that $a^3(b-c)+b^3(c-a)+c^3(a-b)$ contains $b-c$, $c-a$, $a-b$ as factors.

XXXIII. g.

1. Find the quotient and the remainder when $2x^4-3x^3-x^2+x-1$ is divided by $x-3$.

2. Find, to three places of decimals, a positive number such that if it is added to its square, the sum is unity.

3. Two workmen take the same time to earn £22 and £21 respectively. The former earns £15. 8s. in one day less time than the latter takes to earn the same sum. How much does each earn per day?

4. Simplify the expressions

(i) $\left(\dfrac{a^3}{b}-\dfrac{b^3}{a}\right)\left(\dfrac{3a+b}{a+b}-\dfrac{3a-b}{a-b}\right).$

(ii) $\dfrac{1}{(a^2-b^2)(a^2-c^2)}+\dfrac{1}{(b^2-c^2)(b^2-a^2)}+\dfrac{1}{(c^2-a^2)(c^2-b^2)}.$

5. Solve the equations

(i) $\dfrac{a}{x-a}+\dfrac{b}{x-b}=0.$

(ii) $\dfrac{a^2}{x}+\dfrac{b^2}{y}=\dfrac{(a+b)^2}{c}, \quad x+y=c.$

6. A man spends £70 in 45 days; make a graph and read off from it his expenditure in 17, 32, and 41 days, to the nearest pound.

7. If a and β are the roots of the equation $ax^2-bx+c=0$, find the equation whose roots are $2a$ and 2β.

XXXIII. h.

1. Simplify $a^2x^m-b^2x^{m+4}$

2. If the coefficients of x^4 and of x in the product of $2x^3+3x^2+ax-10$ and $3x^3-ax^2-10x+4$ are equal to one another, find the value of a.

3. Find (i) the H.C.F., (ii) the L.C.M. of $a^4+a^2b^2+b^4$, $a^4-a^2b^2+2ab^3-b^4$.

4. In the same diagram draw the graphs of

$$y=x+3, \quad 2y-x=8, \quad \text{and } 2y+5x=20.$$

What do you deduce as to the roots of the different pairs of equations?

5. If a, β are the roots of $x^2-px-q=0$, form the equation whose roots are $-3a$, -3β.

6. Solve the equations (i) $(2x^2+3x-1)(2x^2+3x-2)=156$,
(ii) $2(x-1)(y-1)=6(x+y)=-3xy$.

7. The difference in the average rates of two trains is 13 miles per hour. The faster of the two takes 2 hours less time to travel 164 miles than the slower takes to travel 168 miles. Find their respective rates.

XXXIII. k.

1. If $\frac{x}{y}+\frac{y}{x}=a$, $\frac{y}{z}+\frac{z}{y}=b$, $\frac{z}{x}+\frac{x}{z}=c$, prove that $a^2+b^2+c^2-abc=4$.

2. Solve the equation $4x^2+2x-1=0$, giving results correct to two decimal places.

3. Simplify $\left(\frac{b-c}{a+b-c}-\frac{a-b+c}{c-b}\right)\left(\frac{1}{a}-\frac{c-b}{a^2}\right)$.

4. The denominator of a certain fraction exceeds its numerator by one. Two other fractions are formed, one of them by adding 9 to the denominator, and the other by subtracting 6 from the numerator, of the original fraction. These two fractions are equal. Find the original fraction.

5. An old clock increased uniformly in value from £4. 10s. in the year 1890, to £8. 10s. in 1899. Find graphically its value in 1893, 1894, and 1897, to the nearest shilling.

6. Solve the equations $x^2+y^2=2(a^2+b^2)$, $\frac{(x+y)^2}{a^2}+\frac{(x-y)^2}{b^2}=8$.

7. Construct an equation whose roots shall exceed by a quantity m the roots of the equation $ax^2+bx+c=0$.

XXXIII. l.

1. Resolve into factors (i) $a^4-8a^2b-48b^2$, (ii) $(a^2+b^2)c+(b^2+c^2)a$.

2. Multiply $a^3+4a^2b+8ab^2+8b^3$ by $a^3-4a^2b+8ab^2-8b^3$.

3. Show that if $a+b+c+d=0$, then
$$a^2-b^2+c^2-d^2=2(a+b)(a+d).$$

4. Find the area of the quadrilateral formed by joining the points (10, 20), (13, 9), (23, 8), (28, 20).

5. Solve the equations $x+y+z=6$, $4x+y=2z$, $x^2+y^2+z^2=14$.

6. If a, b, c are real quantities, determine the condition that the roots of the equation $ax^2+2bx+c=0$ may be imaginary.

7. The journey between two towns by one route consists of 233 miles by rail followed by 126 miles by sea; by another route it consists of 405 miles by rail, followed by 39 miles by sea. If the time occupied on the journey is 50 minutes longer by the first route than by the second, find the average speed by rail, assuming it to be the same by each route, and 25 miles an hour faster than the average speed by sea.

XXXIII. m.

1. Simplify $\frac{1}{a-b}\left\{\frac{(a-b)^3+(b-c)^3}{a-c}-(a+c-2b)^2\right\}$.

2. Resolve into factors (i) $18x^2+53x-35$.
(ii) $a^2+2bc-(c^2+2ab)$.
(iii) $(x-3b)^3-4b^2x+12b^3$.

3. Divide $x^6+6x^5-2x^4+37x^3-5x^2+13x-15$ by x^2-x+5, using the method of detached coefficients.

4. Find the value of $\sqrt{13}$ correct to two decimal places by any graphical or geometrical method.

5. Solve the equations (i) $\dfrac{x^2}{y}+\dfrac{y^2}{x}=\dfrac{3}{2}$, $x+y=1$.

(ii) $ab(x^2+1)=x(a^2+b^2)$.

6. Prove that if the roots of the equation $ax^2+2bx+c=0$ are imaginary, the roots of the equation $ax^2+2(a+b)x+a+2b+c=0$ are also imaginary.

7. The marks of a form ranged from 325 to 259. Draw a graph to scale them from 80 to 0, and read off the scaled marks corresponding to the following actual marks gained : 280, 295, 312. Verify one of your results.

XXXIII. n.

1. Find the relation between the constants when the three equations

$$ax+by=c,\quad bx+ay=d,\quad x^2+y^2=xy$$

are simultaneously true.

2. If $f(n)=\dfrac{n(n-1)}{2}$, and $\phi(n)=\dfrac{n(n+1)}{2}$, find the value of

(i) $f(n+1)-\phi(n)$, (ii) $[f(n+1)]^3-[\phi(n-1)]^3$.

3. Find the L.C.M. of $3x^2-4x-4$ and $4x^3-8x^2-x+2$.

4. Find graphically the maximum value of $6x-x^2-\frac{11}{2}$. Verify your result by algebra.

5. A merchant beginning business with a certain capital succeeded in doubling it, but afterwards lost £1000. He employed the remainder in a venture which brought him in a profit of 35 per cent., after which his capital was found to be £10 more than his original capital. Find the amount of that capital.

6. Solve the equations (i) $\dfrac{x^2-(a+b)x-bc}{x-b}=\dfrac{x^2-(a+c)x-bc}{x-c}$

(ii) $ay^2+bxy=b$, $bx^2+axy=a$.

7. If α and β are the roots of the equation $ax^2+bx+c=0$, find the equation whose roots are $\dfrac{1+\alpha}{\beta}$, $\dfrac{1+\beta}{\alpha}$.

XXXIII. p.

1. Find the L.C.M. of x^4+x, x^4-x^2, x^5-x^3, and x^5+x^3+x.

2. Find the quotient when $x^3-y^3+z^3+3xyz$ is divided by $x-y+z$.

3. Multiply $4x^3+3x^2-7$ by $2x^3-x-5$, using the method of detached coefficients.

4. Draw the graph of $y=x^2+2x$, and hence solve the equation

$x^2+2x-7=0$. (Use a large x unit.)

5. Solve the equations (i) $\dfrac{1+2x-3x^2}{1-2x+3x^2}=\dfrac{3-2x+x^2}{3+2x-x^2}$.

(ii) $x^2-y=y^2-x=1\frac{5}{16}$.

6. A and B start in a long-distance race. For 15 minutes A goes at the rate of x yards per second, and B at the rate of $2x$ miles per hour, and then A is leading by 100 yards. Find the value of x.

7. If α, β are the roots of $x^2+px-q=0$, and γ, δ those of $x^2+px+r=0$, prove that $(\alpha-\gamma)(\alpha-\delta)=(\beta-\gamma)(\beta-\delta)=q+r$.

XXXIII. q.

1. Show that $\dfrac{(a+b)^3-c^3}{a+b-c}+\dfrac{(b+c)^3-a^3}{b+c-a}+\dfrac{(c+a)^3-b^3}{c+a-b}$ is equal to $2(a+b+c)^2+a^2+b^2+c^2$.

2. Solve the equations: (i) $ax+by=xy=cx+dy$.

(ii) $\left(\dfrac{x-a}{x+b}\right)^3=\dfrac{x-2a-b}{x+a+2b}$.

3. If $x=\dfrac{ab-cd}{(a-b)-(c-d)}$, show that $\dfrac{x+a}{x-b}=\dfrac{(a-c)(a+d)}{(b-d)(b+c)}$.

4. Find the L.C.M. of $8x^3+27$, $16x^4+36x^2+81$, $6x^2-5x-6$.

5. Draw enough of the graph of $y=x^2$ to enable you to find the square root of 95.

6. A dealer bought 200 sheep. He sold 80 of them so as to gain 4 per cent. on them, and the rest so as to gain $7\frac{1}{2}$ per cent. on them. His whole profit amounted to £21. 7*s*. What did he pay for each sheep?

7. Prove $x^3-px^2+qx-r=0$ to be the equation that results from the elimination of y and z from
$$x+y+z=p,$$
$$xy+yz+zx=q,$$
$$xyz=r.$$

XXXIII. r.

1. Find the factors of each of the following expressions:

$$x^2-1,\ x^2-6x-7,\ x^3-3x^2+2x,\ 3x^2-7x+2.$$

What is their L.C.M.?

2. Simplify (i) $(2x+3)(3x-1)+(2x-5)(5x-3)-(4x-3)^2$.

(ii) $\{(3a+2b)^2-(2a+b)^2\}\div\{7a-2b-(2a-5b)\}$.

3. Draw the graph of $y=x^2-3x$, and hence solve the quadratic $x^2-3x=14$. (Use a large x unit.)

4. Find the condition that $x^2+ax+b^2=0$, and $x^2-bx+a^2=0$ may have a common root.

5. In an election, if one-tenth of those who voted for A had refrained from voting, B would have been returned by a majority of 128, while if one-fifth of those who voted for B had transferred their votes to A, the latter would have been elected by a majority of 535. Which candidate was elected, and by what majority?

6. Solve the equations
$$x(x-y)=10,$$
$$y(x+y)=24.$$

7. If $x+y+z=a$, $x^2+y^2+z^2=b$, $x^3+y^3+z^3=c$, find the product xyz in terms of a, b, c.

XXXIII. s.

1. Prove that $a+b+c$ is a factor of $a^3+b^3+c^3-3abc$.
Deduce the fact that $x+y+z$ is a factor of the expression

$$(x+y)^3+(y+z)^3+(z+x)^3-3(x+y)(y+z)(z+x).$$

2. Solve the equation $(a+b)(ax+b)(a-bx)=(a^2x-b^2)(a+bx)$.

3. If $f(n)=\dfrac{n(n+1)(2n+1)}{6}$, find the value of

(i) $f(n)-f(n-1)$.

(ii) $f(n)-f(n-2)$.

4. If α and β be the roots of the equation $x^2-px+q=0$, form the equation whose roots are $m\alpha^2+n\beta^2$, and $m\beta^2+n\alpha^2$.

5. Find the limits of value between which x must lie in order that $4x^2+4x-35$ may be positive.

6. Solve the equations
$$x+y+z=1,$$
$$x^2+y^2+z^2=9,$$
$$x^3+y^3+z^3=1.$$

7. A and B start from the same place at the same time. After an hour and a quarter A is found to be $7\frac{1}{2}$ miles ahead of B. If, however, A's rate of cycling had been greater by one seventh, and B's by one-fifth, A would have been 8 miles ahead. Find their rates of cycling.

PART II.

*CHAPTER XXXIV.

184. We will here prove the various Laws employed in Algebra, the truth of which has been assumed in previous chapters.

Commutative Law.

CORRIGENDA.

Page 404, XLVIII. a. 5, *for* $\overline{n-7}$ *read* 7.
" " " 6, insert n. before $\overline{n-1}$.

Answers.

Page lviii. XXXIX. b. 15, *for* 0 *read* 1.
" " XXXIX. c. 35, *for* 2·448 *read* 2·45.
" lxv. XLVIII. a. 24, delete 60.
" " " 26, *for* 7620 *read* 6720.

Elementary Algebra, Part II

Again $a \times b \div c$ means that a has to be made b times as great and to be divided into c equal parts. But if a be first divided into c equal parts, and if then each part be made b times as great, the result is the same as before

$$\therefore\ a \times b \div c = (a \div c) \times b.$$

(2) *For Fractions.*

Let a, b, c, d be positive integers.

The expression $\frac{a}{b}$ means that the unit has been divided into b equal parts, of which a parts are taken.

When, therefore, we multiply $\frac{a}{b}$ by $\frac{c}{d}$, we mean that $\frac{a}{b}$ is divided into d equal parts, each equal to $\frac{a}{bd}$, and that c of these parts are taken.

$$\frac{a}{b} \times \frac{c}{d} = \frac{a}{bd} \times c = \frac{ac}{bd}.$$

It follows by similar reasoning that

$$\frac{a}{b} = \frac{a}{b} \times \frac{c}{c} = \frac{ac}{bc}.$$

Associative Law.

185. *In addition and subtraction, we may group (bracket) the terms of an expression in any way.*

$$\begin{aligned} a+b-c-d &= a+(b-c)-d \\ &= a-c+(b-d) \\ &= (a+b)-(c+d) = a-(c-b)-d \text{ and so on (See Art. 14)} \end{aligned}$$

In the same way, in multiplication and division,

$$\begin{aligned} abcd \div e &= a \times (bc) \times d \div e \\ &= \frac{abc}{e} \times d = \frac{a}{e} \times bcd \text{ and so on.} \end{aligned}$$

186. The **Law of Indices** is a particular case of the above.

Thus, $\quad a^2 \times a^3 = aa \times aaa = a^5 = a^{2+3}.$

$$a^5 \div a^3 = (aaaaa) \div (aaa) = \frac{aaaaa}{aaa} = aa = a^2 = a^{5-3}.$$

Distributive Law.

187. If a and b have any values whatever, and c is a positive integer, $(a+b)c = (a+b)+(a+b)+\dots$ taken c times

$$\begin{aligned} &= a+a+a+\dots \quad \dots\dots\dots\dots\dots \\ &\quad +b+b+b+\dots \quad \dots\dots\dots\dots\dots \\ &= ac+bc. \quad \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots (1) \end{aligned}$$

Also if d is any positive integer we see that again

$$(a+b) \div d = a \div d + b \div d,$$

for if we multiply each of these expressions by d, the results are equal.

$$\therefore \quad (a+b)\times c\div d=[(a+b)\times c]\div d$$
$$=(ac+bc)\div d$$
$$=ac\div d+bc\div d,$$

or
$$(a+b)\times\frac{c}{d}=a\times\frac{c}{d}+b\times\frac{c}{d}.$$

$\therefore$ the law illustrated in (1) is shown to be true whether c be a whole number or a fraction, as long as c is positive.

Again, if $$(a+b)c=ac+bc,$$
$$(a+b)(-c)=-(a+b)c \text{ by the Law of Signs}$$
$$=-(ac+bc)$$
$$=-ac-bc$$
$$=a(-c)+b(-c);$$

$\therefore$ for all values of c $\quad (a+b)c=ac+bc.$

In the case of division,
$$(a+b)\div c=(a+b)\times\frac{1}{c}$$
$$=a\times\frac{1}{c}+b\times\frac{1}{c}$$

where c is any quantity, positive or negative (proved above);
$$\therefore \quad (a+b)\div c=a\div c+b\div c.$$

Also
$$\frac{a}{b}+\frac{c}{d}=\frac{ad}{bd}+\frac{bc}{bd}$$
$$=ad\div bd+bc\div bd$$
$$=\frac{ad+bc}{bd}.$$

188. *If an expression has a factor, any multiple of the expression is divisible by that factor.*

Let P be a factor of A, then P will be a factor of mA.

For suppose A $=a$P, then mA $=ma$P.

$\therefore$ P is a factor of mA, *i.e.* mA is divisible by P.

189. *If two expressions have a common factor, the sum and the difference of any multiples of those expressions are divisible by that factor.*

Let P be a common factor of A and B, we have to prove that P is a factor of mA $\pm$ nB.

Let $A = aP$, $B = bP$.

Then $$mA \pm nB = maP \pm nbP = (ma \pm nb)P.$$

$\therefore$ P is a factor of $mA \pm nB$.

190. *To prove the rule for finding the* H.C.F. *of two algebraic expressions.*

Let A, B be the two expressions, from which it is supposed that monomial factors have been removed; and let them be arranged in descending powers of some common letter.

Divide A by B, supposing that A is not of lower degree in that letter than B. Let the quotient be p and remainder C.

Divide B by C. Let the quotient be q and remainder D.

Divide C by D and let the remainder be zero.

The work is shown below.

$$\begin{array}{l} B)\,A\,(p \\ \quad\underline{pB} \\ \quad\;\; C)\,B\,(q \\ \qquad\;\underline{qC} \\ \qquad\;\; D)\,C\,(r \\ \qquad\quad\;\underline{rD} \end{array}$$

From this we see that

$$A - pB = C \;\ldots\ldots\ldots (1)$$
$$B - qC = D \;\ldots\ldots\ldots (2)$$
$$rD = C \;\ldots\ldots\ldots (3)$$

To show that D is *a* common factor of A and B.

From (1) $A = pB + C$
$= p(qC + D) + C \ldots$ from (2)
$= p(qrD + D) + rD \ldots$ from (3)
$= (pqr + p + r)D$;

$\therefore$ D is a factor of A.

From (2) $B = qC + D = qrD + D = (qr + 1)D$;

$\therefore$ D is a factor of B.

Thus D is a common factor of A and B.

Next, to show that it is the *highest* common factor.

All the common factors of A and B divide $A - pB$, *i.e.* C.

Thus every common factor of A and B divides $B - qC$, *i.e.* D.

Thus D is *a* common factor, and all the other common factors divide it;

$\therefore$ D is the H.C.F.

Monomial factors were removed from A and B;

$\therefore$ D contains no such factors.

Moreover, if at any stage a monomial factor were introduced for convenience into *either* the divisor or the dividend (say D or C), it would not appear in the result; for it would not be a common factor of D and C, and consequently not a common factor of A and B.

Also a monomial factor might at any stage be removed without affecting the result; for such factors do not appear in D.

191. *The* L.C.M. *of two expressions is equal to the product of the expressions, divided by their* H.C.F.

Or, *The* L.C.M. *of two expressions multiplied by their* H.C.F. *is equal to the product of the two expressions.*

Let L be the L.C.M., and H the H.C.F. of A and B.

Let $A = aH$, $B = bH$, so that, by hypothesis *a and b have no common factor.*

$\therefore$ The L.C.M. of A and B is the L.C.M. of aH and bH, which is abH, which $= \frac{AB}{H}$.

*CHAPTER XXXV.

HARDER FACTORS. SYMMETRICAL EXPRESSIONS. CYCLIC ORDER.

192. *For all positive integral values of n, the expression* $x^n - a^n$ *is divisible by* $x - a$.

When we divide $x^n - a^n$ by $x - a$, the remainder $= a^n - a^n = 0$. (Remainder Theorem, Art. 95.)

$\therefore$ $x^n - a^n$ is divisible by $x - a$, as long as n is integral and positive.

We have already used simple cases of this,

$$x^2 - a^2 = (x-a)(x+a), \quad (x^3 - a^3) = (x-a)(x^2 + ax + a^2).$$

Again, n being a positive integer, the remainder, when we divide $x^n + a^n$ by $x - a$, is $a^n + a^n = 2a^n$; $\therefore$ $x^n + a^n$ is never divisible by $x - a$ when n is a positive integer.

If n be a positive **even** *integer,* $x^n - a^n$ *is divisible by* $x + a$.

When we divide $x^n - a^n$ by $x + a$, the remainder is $(-a)^n - a^n$.

Now if n be an *even* integer $(-a)^n = a^n$;

$\therefore$ the remainder is zero;

$\therefore$ $x^n - a^n$ is then divisible by $x+a$.

If n be an *odd* integer, the remainder

$$=(-a)^n - a^n = -a^n - a^n = -2a^n,$$

which is not zero; $\therefore$ when n is an *odd* positive integer, $x^n - a^n$ is not divisible by $x+a$.

193. *If n be any* **odd** *positive integer, the expression $x^n + a^n$ is divisible by $x+a$.*

The remainder, when we divide $x^n + a^n$ by $x+a$, is $(-a)^n + a^n$.

Now if n be an *odd* positive integer, $(-a)^n = -a^n$;

$\therefore$ the remainder is zero;

i.e. $x^n + a^n$ is then divisible by $x+a$.

If n be an *even* positive integer, the remainder $= a^n + a^n = 2a^n$, which is not zero; $\therefore$ $x^n + a^n$ is not divisible by $x+a$ in that case.

194. The theory of quadratic equations is often useful in the determination of factors.

Example 1. Consider the expression $3x^2 - 5x + 7$.

If we solve the equation $3x^2 - 5x + 7 = 0$,

the quantity under the radical sign $= 25 - 4 \times 3 \times 7 \qquad (b^2 - 4ac)$

$= 25 - 84$,

which is negative; $\therefore$ the equation has imaginary roots;

$\therefore$ the expression $3x^2 - 5x + 7$ has no real factors.

Example 2. Find factors of the expression $x^2 - 4x + 1$.

If we solve the equation $x^2 - 4x + 1 = 0$,

$$x = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= \frac{4 \pm \sqrt{12}}{2}$$

$$= 2 \pm \sqrt{3};$$

$\therefore$ the reqd. factors are $x - 2 - \sqrt{3}$ and $x - 2 + \sqrt{3}$.

The method of finding factors by completing squares is useful.

$$x^2 - 4x + 1$$
$$= x^2 - 4x + 4 - 3$$
$$= (x-2)^2 - 3$$
$$= (x - 2 + \sqrt{3})(x - 2 - \sqrt{3}).$$

195. $\mathbf{a^3 + b^3 + c^3 - 3abc = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)}$.

This can be seen by dividing $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$. (See Ex. 2, Art. 93.)

By writing $-b$ instead of b in the above we see that

$$a^3 - b^3 + c^3 + 3abc = (a - b + c)(a^2 + b^2 + c^2 + ab + bc - ca).$$

In the same way, writing $-a$ for a,

$$-a^3+b^3+c^3+3abc=(-a+b+c)(a^2+b^2+c^2+ab-bc+ca),$$

and similarly

$$a^3+b^3-c^3+3abc=(a+b-c)(a^2+b^2+c^2-ab+bc+ca).$$

Example 1.

$$\begin{aligned} & x^3+8y^3+z^3-6xyz \\ &= x^3+(2y)^3+z^3-3(2x)yz \\ &= (x+2y+z)(x^2+4y^2+z^2-2xy-2yz-zx). \end{aligned}$$

Example 2. Factorize the expression

$$(x^2-3x+4)^2-5x(x^2-3x+4)+6x^2.$$

The expression is in the form $a^2-5ax+6x^2$ which is equal to

$$(a-2x)(a-3x);$$

$$\begin{aligned} \therefore \text{ the expression} &= (x^2-3x+4-2x)(x^2-3x+4-3x) \\ &= (x^2-5x+4)(x^2-6x+4) \\ &= (x-1)(x-4)(x^2-6x+4). \end{aligned}$$

Example 3. Factorize the expression $a^4+a^2b^2+b^4$.

$$\begin{aligned} a^4+a^2b^2+b^4 &= (a^4+2a^2b^2+b^4)-a^2b^2 \\ &= (a^2+b^2)^2-a^2b^2 \\ &= (a^2+b^2+ab)(a^2+b^2-ab) \\ &= (a^2+ab+b^2)(a^2-ab+b^2). \end{aligned}$$

Example 4. Factorize the expression

$$x^4-5x^2-12-\frac{5}{x^2}+\frac{1}{x^4}.$$

$$\begin{aligned} \text{The given expression} &= x^4+\frac{1}{x^4}-5\left(x^2+\frac{1}{x^2}\right)-12 \\ &= \left(x^2+\frac{1}{x^2}\right)^2-2-5\left(x^2+\frac{1}{x^2}\right)-12 \\ &= \left(x^2+\frac{1}{x^2}\right)^2-5\left(x^2+\frac{1}{x^2}\right)-14 \\ &= a^2-5a-14, \text{ writing } a \text{ for } x^2+\frac{1}{x^2} \\ &= (a-7)(a+2) \\ &= \left(x^2-7+\frac{1}{x^2}\right)\left(x^2+2+\frac{1}{x^2}\right) \\ &= \left(x^2-7+\frac{1}{x^2}\right)\left(x+\frac{1}{x}\right)^2. \end{aligned}$$

Example 5. Factorize the expression

$$6x^2+7xy-3y^2-14x-10y+8.$$

Consider the equation $6x^2+7xy-3y^2-14x-10y+8=0$.

Arranging as a quadratic for x,

$$6x^2+x(7y-14)-3y^2-10y+8=0.$$

$$x=\frac{-(7y-14)\pm\sqrt{(7y-14)^2-24(-3y^2-10y+8)}}{12},\quad\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$$

$$=\frac{-7y+14\pm\sqrt{121y^2+44y+4}}{12}$$

$$=\frac{-7y+14\pm(11y+2)}{12}$$

$$=\frac{4y+16}{12}\text{ or }\frac{-18y+12}{12}$$

$$=\frac{y+4}{3}\text{ or }\frac{-3y+2}{2}.$$

$\therefore$ $3x-y-4$ and $2x+3y-2$ are the reqd. factors.

DIVISION BY MEANS OF FACTORIZATION.

196. *Divide* $x^2+2(a+b)x+a^2+2ab+b^2$ *by* $x+a+b$.

$$x^2+2x(a+b)+a^2+2ab+b^2=x^2+2x(a+b)+(a^2+2ab+b^2)$$

$$=(x+\overline{a+b})^2.\quad[(p+q)^2=p^2+2pq+q^2]$$

$\therefore$ $x+a+b$ is the reqd. quotient.

Divide $a^2+b^2-c^2-d^2+2ab-2cd$ *by* $a+b-c-d$.

$$a^2+b^2-c^2-d^2+2ab-2cd=(a^2+2ab+b^2)-(c^2+2cd+d^2)$$

$$=(a+b)^2-(c+d)^2$$

$$=(a+b+c+d)(a+b-c-d).$$

$\therefore$ $a+b+c+d$ is the reqd. quotient.

Divide $x^2+2xy+y^2+x+y-6$ *by* $x+y-2$.

$$x^2+2xy+y^2+x+y-6=(x+y)^2+(x+y)-6$$

$$=(x+y+3)(x+y-2).$$

$$[a^2+a-6=\overline{a+3}\,.\,\overline{a-2}]$$

$\therefore$ $x+y+3$ is the reqd. quotient.

Examples. XXXV. a.

1. Prove, without actual division, that x^6-a^6 is divisible by $x-a$ without remainder. Find three terms of the quotient by division, and hence write down the complete quotient.

Do the same with

2. x^7+a^7 and $x+a$.

3. x^7-a^7 and $x-a$.

4. x^8-a^8 and $x+a$

5. x^8-a^8 and $x-a$.

6. x^6-64 and $x-2$.

7. x^6-64 and $x+2$.

Factorize the following expressions:

8. $x^3+x^2-4x-4.$

9. $(x^2+x)^2-2(x^2+x)-3.$

10. $\frac{27}{x^3}-1.$

11. $a^3+\frac{1}{a^3}.$

12. $(x^2-4x)^2+7(x^2-4x)+12.$

13. $x^2y^2-x^2-y^2+1.$

14. $c(c-b)+a(b-a).$

15. $x^2-2x-2.$

16. $a^4-23a^2+1.$

17. $abx^3+(a^2-b)x^2-1.$

18. $a(a+c)-b(b+c).$

19. $a^3+b^3+1-3ab.$

20. $x^2-10x+23.$

21. $4x^2-4xy+y^2-10x+5y-24.$

22. $x^4+4.$

23. $x^3-8y^3-27-18xy.$

24. $4x^2-8x+1.$

25. $9x^2-16y^2-9x+28y-10.$

26. $15x^2-16xy+4y^2+41x-22y+28.$

27. $x^4+x^2+1.$

28. $a^3-b^3+8+6ab.$

29. $16x^2+8x-5.$

30. $16x^4+4x^2+1.$

31. $8a^3+b^3-1+6ab.$

32. $(2x^2-5x+3)(2x^2-5x+4)-2.$

33. $6x^2-19xy+15y^2+23x-36y+21.$

34. $x^4+64.$

35. $x^4-x^2y^2+16y^4.$

36. $(x^2+2x-8)(x^2+2x-2)-27.$

37. $x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5.$

38. $81a^4+9a^2+1.$

39. $2x^2+5xy+3y^2+6x+10y-8.$

40. $x^4+4x^2+16.$

41. $81x^4+64.$

42. $x^5-ax^4+a^2x^3-a^3x^2+a^4x-a^5.$

43. $x^2+x-10+\frac{1}{x}+\frac{1}{x^2}.$

44. $x^3+11x-6(x^2+1).$

45. $x^2-5x-10-\frac{10}{x}+\frac{4}{x^2}.$

46. $2x^3+5x^2+x-2.$

47. $x^3-5x-\frac{5}{x}+\frac{1}{x^3}.$

48. $x^4+3x^2-8+\frac{3}{x^2}+\frac{1}{x^4}.$

[*The following examples should be done by using your knowledge of factors, and not by actual multiplication and division.*]

Find the product of:

49. $x+2y+3z$ and $x+2y-3z.$

50. $2x-3y+4z$ and $2x+3y+4z.$

51. $a+b-2$ and $a+b-3.$

52. $a+b+x+y$ and $a+b-x-y.$

53. x^2-x+1 and $x^2+x+1.$

54. x^2-x+2 and $x^2+x+2.$

55. x^2-x-2 and $x^2+x+2.$

56. $x^2-3xy+5y^2$ and $x^2+3xy+5y^2.$

57. $2a+b+x+2y$ and $2a+b-x-2y.$

58. $2x^2+3xy-4y^2$ and $2x^2-3xy+4y^2.$

59. x^3+x^2+x+1 and $x^3+x^2-x-1.$

60. x^2-2x+4 and $x^2+2x+4.$

61. $(2a+b)^2-(a+2b)^2$ and $(2a-b)^2-(a-2b)^2.$

62. $x+1+\frac{1}{x}$ and $x-1+\frac{1}{x}.$

63. x^3-2x^2+x-2 and x^3+2x^2+x+2

64. $(x^2+x+1)^2-(x^2-x+1)^2$ and $(2x^2+x-2)^2-(2x^2-x-2)^2.$

65. $2ax-2x^2$, $3ax+3x^2$, $a^4+a^2x^2$.
66. $(a-b)$, $(a+b)$, (a^2+ab+b^2), (a^2-ab+b^2).
67. $(a-b)^2$, $(a+b)^2$, $(a^2+b^2)^2$.
68. $(3x-2)$, $(5x+4)$, $(15x^2-2x-8)$.
69. $(6x^2+5x-4)$, $(3x-4)$, $(2x+1)$.
70. $(x-2)$, $(x+2)$, (x^2-2x+4), (x^2+2x+4).
71. $(a-b)^3$, $(a+b)^3$, $(a^2+b^2)^3$.
72. $(3a+b)^2-(a+3b)^2$, $(3a-b)^2-(a-3b)^2$, $(a^2+b^2)^2$.
73. $(a+b+c)$, $(a+b-c)$, $(a-b+c)$, $(a-b-c)$.

74. Divide $(3x^2+2x+1)^2-(2x^2-x+5)^2$ by x^2+3x-4.
75. Simplify $(6x^2+x-1)\times(2x^2-9x+4)\div(3x^2-13x+4)$.
76. Show that $(2x^2+3x+2)^3-(x^2+x+1)^3$ is divisible by $(x+1)^2$.
77. Divide the product of $2x^2+x-6$ and $3x^2-5x-2$ by $2x^2-7x+6$.
78. Divide $(a-2b+3)^2-(a+2b-5)^2$ by $a-1$.
79. Show that $(3x^2-7x+5)^3+(x^2+3x-4)^3$ is divisible by $(2x-1)^2$.
80. Simplify $[x^2+ax-bx-ab]\times[x^2-ax+bx-ab]\div[x^2-a^2]$.
81. Multiply the square of $3a-2b$ by $9a^2+12ab+4b^2$.

Write down the quotient when

82. x^4-y^4 is divided by $x-y$.
83. x^4-y^4 is divided by $x+y$.
84. x^4-16y^4 is divided by $x-2y$.
85. x^4-16y^4 is divided by $x+2y$.
86. Prove that $x^{140}-a^{140}$ is divisible by $x-a$, by x^2-a^2, and by x^2+a^2
87. Prove that $x^6+3x^4+4x^2+224$ is divisible by x^2+7.

SYMMETRICAL EXPRESSIONS.

197. An expression is said to be *symmetrical* when it is unaltered by interchanging *any* pair of the letters which it contains.

Thus the following are symmetrical expressions:

$$a+b+c, \quad a^3+b^3+c^3,$$

$$a^2+b^2+c^2-ab-bc-ca, \quad ab+bc+cd+da+ac+bd.$$

The expression $(a-b)(b-c)(c-a)$ is not symmetrical, for if we interchange a and b, the expression becomes

$$(b-a)(a-c)(c-b),$$

$$\text{i.e.} \quad -(a-b)(c-a)(b-c).$$

In connection with symmetrical expressions the following notation is useful.

$\sum\limits_{abc}[a(b-c)]=$ the sum of all terms like $a(b-c)$ when we interchange the letters a, b, c, in order;

i.e. $$\sum_{abc}[a(b-c)] = a(b-c)+b(c-a)+c(a-b);$$

$$\sum_{abc}\frac{\quad}{(a-b)(a-c)} = \frac{\quad}{(a-b)(a-c)} + \frac{b^2}{(b-c)(b-a)} + \frac{c^2}{(c-a)(c-b)}.$$

The following are simpler cases:

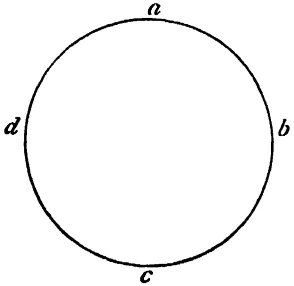
$$\sum_{abc}(a^2) = a^2+b^2+c^2;$$

$$\sum_{abc}(a^2-bc) = (a^2-bc)+(b^2-ca)+(c^2-ab).$$

CYCLIC SYMMETRY. CYCLIC ORDER.

198. When an expression is written so that the letters forming it are taken in a selected unending order (generally the order of the alphabet) it is said to be in *cyclic symmetry*, and the letters are said to be taken in *cyclic order*.

This is well illustrated by arranging the letters in order round the circumference of a circle.



Arranging the letters a, b, c, d in this way and looking at the expression

$$(a-b)(b-c)(c-d)(d-a)$$

we see that the letters are taken in cyclic order, and the expression has cyclic symmetry.

On the other hand, in the expression

$$(a-b)(c-b)(d-c)(d-a)$$

the order is changed in the second and third factors.

By changing the signs in these factors, the expression may be written $$(a-b)(b-c)(c-d)(d-a),$$

and is now reduced to cyclic symmetry.

The work will be much simplified if the expressions used are first arranged in cyclic symmetry, where such is possible.

Dimensions. The **dimensions** of any algebraical term are determined by taking the sum of the indices of the various letters employed.

Thus $9x^2y^3$ is of five dimensions; $17x^3yz^4$ is of eight dimensions. In the latter case it must be remembered that the index of y is unity.

Numerical coefficients are not taken into account.

Homogeneous. A compound algebraical expression is said to be *homogeneous* when all its terms are of the same dimensions. $x^2+2xy+y^2$ is homogeneous, every term being of two dimensions. $a^5-5a^4b+6a^3b^2-5a^2b^3+ab^4-b^5$ is homogeneous, every term being of five dimensions.

HOMOGENEOUS AND SYMMETRICAL EXPRESSIONS.

199. *Find the factors of $ab(a-b)+bc(b-c)+ca(c-a)$.*

Let X denote the given expression.

When $a=b$,

$$\begin{aligned} X &= 0+bc(b-c)+cb(c-b), \\ &= bc(b-c+c-b) \\ &= 0. \end{aligned}$$

$\therefore$ $a-b$ is a factor of X, by the Remainder Theorem.

In the same way we may prove that $b-c$ and $c-a$ are also factors of X.

Also no two of the factors $a-b$, $b-c$, $c-a$ have a common factor,

$\therefore$ X is divisible by $(a-b)(b-c)(c-a)$ without remainder.

Again, X is entirely of the third degree,

and $(a-b)(b-c)(c-a)$ is also of the third degree,

$\therefore$ the only other possible factor of X is a numerical one.

Let that factor be k, so that

$$ab(a-b)+bc(b-c)+ca(c-a)=k(a-b)(b-c)(c-a).$$

This equation must be true for all values of a, b, and c,

$\therefore$ putting $a=0$, $b=1$, and $c=-1$,

$$-1(1+1)=k(-1)(2)(-1),$$

$$k=-1.$$

$$\therefore\ ab(a-b)+bc(b-c)+ca(a-c)=-(a-b)(b-c)(c-a).$$

The value of k might be determined in the following manner

Examine the equation

$$ab(a-b)+bc(b-c)+ca(c-a)\equiv k(a-b)(b-c)(c-a).$$

The coeff. of a^2b on the *left-hand side* is unity.

The coeff. of a^2b on the *right-hand side* is $-k$.

(We see this by taking the product of k, with a in the first factor, b in the second factor, and $-a$ in the third factor.)

$$\therefore \quad -k=1 \quad \text{or } k=-1.$$

The expression $ab(a-b)+bc(b-c)+ca(c-a)$ may be written in any of the following forms

$$a^2b-b^2a+b^2c-c^2b+c^2a-a^2c,$$

$$a(c^2-b^2)+b(a^2-c^2)+c(b^2-a^2),$$

or
$$a^2(b-c)+b^2(c-a)+c^2(a-b).$$

We therefore have the following important results:

$$\begin{aligned} \mathbf{ab(a-b)+bc(b-c)+ca(c-a)} & \\ &\equiv \mathbf{a^2b-b^2a+b^2c-c^2b+c^2a-a^2c} \\ &\equiv \mathbf{a(c^2-b^2)+b(a^2-c^2)+c(b^2-a^2)} \\ &\equiv \mathbf{a^2(b-c)+b^2(c-a)+c^2(a-b)} \\ &\equiv \mathbf{-(a-b)(b-c)(c-a)}, \end{aligned}$$

or

$$\sum_{abc}[\mathbf{ab(a-b)}]\equiv\sum_{abc}[\mathbf{a(c^2-b^2)}]\equiv\sum_{abc}[\mathbf{a^2(b-c)}]\equiv\mathbf{-(a-b)(b-c)(c-a)}.$$

200. *Factorize the expression* $\sum_{abc}[a^3(b-c)]$,

$$i.e. \quad a^3(b-c)+b^3(c-a)+c^3(a-b).$$

Let X denote the given expression.

When $$b=c,$$

$$\mathsf{X}=b^3(b-a)+b^3(a-b)=0;$$

$$\therefore \quad b-c \text{ is a factor of } \mathsf{X}.$$

In the same way it may be proved that $c-a$ and $a-b$ are also factors.

Now X is of the *fourth* degree and $(a-b)(b-c)(c-a)$ is of the *third* degree.

$\therefore$ there will be one more factor, and it must be of the *first* degree.

Also it must be *symmetrical* with regard to a, b, and c.

It must therefore be $k(a+b+c)$ where k is some numerical quantity.

$$\therefore \quad a^3(b-c)+b^3(c-a)+c^3(a-b)\equiv k(a-b)(b-c)(c-a)(a+b+c) \ldots (1)$$

Putting $a=0,\ b=1,\ c=2,$

$$2+8(-1)=k(-1)(-1)(2)(3)$$

whence $$k=-1;$$

$$\therefore\ \sum_{abc}[a^3(b-c)]\equiv-(a-b)(b-c)(c-a)(a+b+c).$$

k might be determined as follows.

The coefft. of a^3b on the left in (1) is unity.

.... right ...... is $-k$.

(This is found by taking a from the first and fourth factors, $-a$ from the third, and b from the second.)

$$\therefore\ -k=1,\ \text{i.e. } k=-1.$$

201. *Find the factors of*

$$\sum_{abc}[a^2(b-c)^3],\ \text{ i.e. }\ a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3.$$

Let X denote the given expression.

When $a-b=0,\quad \mathsf{X}=a^2(a-c)^3+a^2(c-a)^3$

$$=0,\text{ for }(c-a)^3=[-(a-c)]^3=-(a-c)^3.$$

$$\therefore\ a-b \text{ is a factor of X}.$$

In the same way it may be proved that

$$b-c \text{ and } c-a \text{ are factors.}$$

$$\therefore\ \mathsf{X}\text{ is divisible by }(a-b)(b-c)(c-a).$$

Now this expression is of the *third* degree, and X is of the *fifth* degree, therefore the remaining factor must be of the *second* degree.

Moreover it must be *symmetrical* with regard to a, b, and c.

$\therefore$ it must be of the form

$$k(a^2+b^2+c^2)+l(ab+bc+ac),$$

where k and l are numerical quantities.

$$\therefore\ a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3$$

$$\equiv(a-b)(b-c)(c-a)[k(a^2+b^2+c^2)+l(ab+bc+ca)].$$

When $a=0,\ b=1,\ c=2,$

$$(2)^3+4(-1)^3=(-1)(-1)(2)[k(1+4)+l(2)]$$

$$\text{i.e. } 5k+2l=2. \qquad\qquad (1)$$

When $a=0, b=-1, c=2,$

$$(2)^3+4(1)^3=(1)(-3)(2)[k(1+4)+l(-2)]$$

$$\text{i.e. } 5k-2l=-2. \qquad\qquad (2)$$

To find k and l we must solve equations (1) and (2).

By addition $\qquad k = 0.$

By subtraction $\qquad l = 1.$

$$\therefore \sum_{abc}[a^2(b-c)^3] = (a-b)(b-c)(c-a)(ab+bc+ca).$$

Methods of testing accuracy of working—

(1) In dealing with homogeneous expressions it is useful to notice that if two such expressions are multiplied together, or if one is divided by the other, the result must be homogeneous.

Thus if an expression in which every term is of the 5th degree be multiplied by one in which every term is of the 2nd degree, the product must be homogeneous of the 7th degree. Any term which does not stand this test betrays an error.

Similarly, if the former expression be divided by the latter, the quotient must have all its terms of the 3rd degree.

(2) Results may be checked by substituting special values for the letters employed.

(3) In multiplying together symmetrical expressions the product must be symmetrical.

202. The equation $x^2+y^2=0$ may be interpreted geometrically. We know that the equation $x^2+y^2=c^2$ represents a circle whose centre is at the point (0, 0) and whose radius $=c$.

By making c zero we see that the equation $x^2+y^2=0$ represents a circle whose centre is at the point (0, 0) and whose radius $=0$, *i.e.* it represents the points (0, 0).

$\therefore$ the equation $x^2+y^2=0$ is equivalent to the statement that $x=0$ *and* $y=0$.

From this we see that, if an equation can be reduced to the statement that the sum of certain squares is zero, we can equate these squares simultaneously to zero.

Examples. XXXV. b.

Write down in full:

1. $\sum\limits_{abc} (a^3 - bc).$ 2. $\sum\limits_{xyz} \frac{yz}{(y+x)(z+x)}.$ 3. $\sum\limits_{abc} \frac{a^3 - bc}{(b-a)(c-a)}.$

4. $\sum\limits_{abc} \frac{x^2 - a^2}{(a-b)(a-c)}.$ 5. $\sum\limits_{abc} [(a-b)(a-c)(x+a)].$

Find the factors of the following expressions:

6. $\sum\limits_{abc} (a-b)^3.$ 7. $\sum\limits_{abc} [a^2(b^3-c^3)].$ 8. $\sum\limits_{xyz} [x^4(y^2-z^2)].$ 9. $\sum\limits_{abc} [bc(b^3-c^3)].$

Prove the following identities:

10. $\sum\limits_{abc} [a^2b^2(a-b)] \equiv -(a-b)(b-c)(c-a)(ab+bc+ca).$

11. $\sum\limits_{xyz} [x^4(y-z)] \equiv -(x-y)(y-z)(z-x)(x^2+y^2+z^2+xy+yz+zx).$

12. $\sum\limits_{abc} [a(b-c)^3] \equiv (a-b)(b-c)(c-a)(a+b+c).$

13. $\sum\limits_{abc} (a-b)^5 \equiv 5(a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca).$

14. $\sum\limits_{abc} [(b-c)(x-b)(x-c)] = -(a-b)(b-c)(c-a).$

15. $(x-y)(x+y)^2+(y-z)(y+z)^2+(z-x)(z+x)^2+(x-y)(y-z)(z-x) \equiv 0.$

16. $(a+b+c)^3+(a-b-c)^3+(b-c-a)^3+(c-a-b)^3 \equiv 24abc.$

17. $(x-y)(x+y)^3+(y-z)(y+z)^3+(z-x)(z+x)^3$
$$+2(x-y)(y-z)(z-x)(x+y+z) \equiv 0.$$

18. $(x+y+z)^5+(x-y-z)^5+(y-z-x)^5+(z-x-y)^5 \equiv 80xyz(x^2+y^2+z^2).$

Reduce the following to their simplest forms:

(*Check your results by giving a, b, c, etc., particular values.*)

19. $\frac{a+b}{ab+c^2-ac-bc}+\frac{a+c}{ac+b^2-ab-bc}+\frac{b+c}{bc+a^2-ab-ac}.$

20. $\frac{c}{ab(b-c)(c-a)}+\frac{b}{ac(a-b)(b-c)}+\frac{a}{bc(a-b)(c-a)}.$

$\frac{x(y+z)}{(x-y)(z-x)}+\frac{y(z+x)}{(x-y)(y-z)}+\frac{z(x+y)}{(z-x)(y-z)}.$

22. $\frac{a^2(b-c)^3+b^2(c-a)^3+c^2(a-b)^3}{bc+ca+ab}.$

23. $\frac{b-c}{a^2-(b-c)^2}+\frac{c-a}{b^2-(c-a)^2}+\frac{a-b}{c^2-(a-b)^2}.$

24. $\frac{1}{x(x-y)(x-z)}+\frac{1}{y(y-z)(y-x)}+\frac{1}{z(z-x)(z-y)}.$

25. $\frac{x^2}{(a-x)(x-y)(x-z)}+\frac{y^2}{(a-y)(y-x)(y-z)}+\frac{}{(a-z)(z-x)(z-y)}.$

26. $\dfrac{2(b-c)(c-a)(a-b)}{(b+c)(c+a)(a+b)}-\dfrac{b^2-c^2}{(c+a)(a+b)}-\dfrac{c^2-a^2}{(a+b)(b+c)}-\dfrac{a^2-b^2}{(b+c)(c+a)}$.

27. $\dfrac{a^3(b^2-c^2)+b^3(c^2-a^2)+c^3(a^2-b^2)}{a^2(b-c)+b^2(c-a)+c^2(a-b)}$.

28. $\sum\limits_{abc}\dfrac{1}{(a-b)(a-c)}$. 29. $\sum\limits_{abc}\dfrac{1}{a(a-b)(a-c)}$. 30. $\sum\limits_{abc}\dfrac{b+c}{(c-a)(a-b)}$.

31. $\sum\limits_{abc}\dfrac{1}{a^2(a^2-b^2)(a^2-c^2)}$. 32. $\sum\limits_{abc}\dfrac{a^2}{(a-b)(a-c)}$. 33. $\sum\limits_{abc}\dfrac{a^3}{(c-a)(a-b)}$.

34. $\sum\limits_{abc}\dfrac{1}{(a-b)(a-c)(x-a)}$. 35. $\sum\limits_{abc}\dfrac{a}{(a-b)(a-c)(x-a)}$.

36. $\sum\limits_{abc}\dfrac{a^2}{(a-b)(a-c)(x+a)}$. 37. $\sum\limits_{abc}\dfrac{a(b+c)}{(a-b)(a-c)}$.

38. $\sum\limits_{abc}\dfrac{a^2-bc}{(a+b)(a+c)}$. 39. $\sum\limits_{xyz}\dfrac{x^2y^2}{(z^2-x^2)(z^2-y^2)}$.

203. *Find the value of λ if the expression $x^2-xy-6y^2+7x-y+\lambda$ can be resolved into two factors of the first degree.*

Taking the terms of the second degree $x^2-xy-6y^2$, we see that they can be resolved into $(x-3y)(x+2y)$.

$\therefore$ the factors of the given expression will be of the form

$$(x-3y+a)(x+2y+b).$$

Multiplying, this expression is equal to

$$x^2-xy-6y^2+(a+b)x+(2a-3b)y+ab.$$

Hence by comparison with the given expression we see that

$$a+b=7,\quad 2a-3b=-1,\quad \text{and}\quad ab=\lambda.$$

From the first two of these equations

$$a=4,\quad \text{and}\quad b=3,\quad \therefore\ \lambda=ab=12.$$

204. *If the expression $xy+gx+fy+c$ can be resolved into two factors of the first degree, find the relation between g, f, and c.*

If the expression has factors they must evidently be of the form $(x+\alpha)$, $(y+\beta)$.

$\therefore$ their product $xy+\beta x+\alpha y+\alpha\beta$ must be the same as the given expression.

$\therefore$ by comparison,

$$\beta=g,\quad \alpha=f,\quad \text{and}\quad \alpha\beta=c.$$

$$\therefore\ fg=c.$$

205. *If two expressions of the n^{th} degree (n being a positive integer) are identically equal to one another, the coefficient of any power of x in one expression is equal to the coefficient of the same power of x in the other.*

Let the two expressions be

$$ax^n + bx^{n-1} + \ldots + qx^2 + rx + s,$$

and $$a'x^n + b'x^{n-1} + \ldots + q'x^2 + r's + s'.$$

They are equal to one another for all values of x.

$\therefore$ putting x equal to zero, $s = s'$.

$$\therefore \ ax^n + bx^{n-1} + \ldots + qx^2 + rx \equiv a'x^n + b'x^{n-1} + \ldots + q'x^2 + r'x.$$

Dividing both sides by x,

$$ax^{n-1} + bx^{n-2} + \ldots + qx + r \equiv a'x^{n-1} + b'x^{n-2} + \ldots + q'x + r'.$$

Putting $x = 0$ in this identity,

$$r = r'.$$

In the same way we can prove that $q = q' \ldots b = b'$, and $a = a'$.

206. *To find the relations between the coefficients in order that the expression $x^4 + ax^3 + bx^2 + cx + d$ may be a perfect square.*

If it is a square it must be of the form $(x^2 + px + \sqrt{d})^2$.

$$\therefore \ x^4 + ax^3 + bx^2 + cx + d \equiv (x^2 + px + \sqrt{d})^2$$
$$\equiv x^4 + 2px^3 + (2\sqrt{d} + p^2)x^2 + 2px\sqrt{d} + d.$$

$\therefore$ by equating coefficients of like powers of x we get

$$2p = a, \quad \ldots\ldots\ldots\ldots(1)$$
$$2\sqrt{d} + p^2 = b, \quad \ldots\ldots\ldots\ldots(2)$$
$$2p\sqrt{d} = c. \quad \ldots\ldots\ldots\ldots(3)$$

$\therefore$ from (1) and (3) by divison $\sqrt{d} = \dfrac{c}{a}$.

$$\therefore \ a^2d = c^2. \quad \ldots\ldots\ldots\ldots(4)$$

From (2) $$8\sqrt{d} = 4b - 4p^2$$
$$= 4b - a^2 \ldots \text{from (1)}.$$

$\therefore$ by squaring we get

$$64d = (4b - a^2)^2. \quad \ldots\ldots\ldots\ldots(5)$$

Equations (4) and (5) are the required conditions.

207. *If* α, β, γ *are the roots of the equation* $x^3+px^2+qx+r=0$, $\alpha+\beta+\gamma=-p$, $\alpha\beta+\beta\gamma+\gamma\alpha=q$, $\alpha\beta\gamma=-r$.

α, β, γ evidently satisfy the equation $(x-\alpha)(x-\beta)(x-\gamma)=0$. This may be written

$$x^3-(\alpha+\beta+\gamma)x^2+(\alpha\beta+\beta\gamma+\gamma\alpha)x-\alpha\beta\gamma=0. \quad \text{.........(1)}$$

Thus we see that the given equation and equation (1) must be identically the same.

$\therefore$ by comparison

$$\alpha+\beta+\gamma=-p, \quad \alpha\beta+\beta\gamma+\gamma\alpha=q, \quad \text{and} \quad \alpha\beta\gamma=-r.$$

Similar properties may be proved for equations of higher degrees.

The sum of the squares of the roots

$$=(\alpha+\beta+\gamma)^2-2(\beta\gamma+\gamma\alpha+\alpha\beta)=p^2-2q.$$

208. *If two expressions of the* n^{th} *degree in* x *are equal for more than* n *values of* x, *they are identical; i.e. they are equal for all values of* x.

Suppose that $ax^n+bx^{n-1}+cx^{n-2}+\ldots=px^n+qx^{n-1}+rx^{n-2}+\ldots$, (1) when x has the values $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ and β.

Then $\quad (a-p)x^n+(b-q)x^{n-1}+(c-r)x^{n-2}+\ldots=0, \quad$ ....(2) when x has these values.

$\therefore$ by the Remainder Theorem, $x-\alpha_1$, $x-\alpha_2$, $x-\alpha_3 \ldots x-\alpha_n$ are all factors of the left-hand side of equation (2).

$$\therefore \ (a-p)x^n+(b-q)x^{n-1}+(c-r)x^{n-2}+\ldots$$
$$=(a-p)(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)$$

Now, by hypothesis, the left-hand side of this equation vanishes when $x=\beta$. $\therefore$ the right-hand side also vanishes when $x=\beta$. $\quad \therefore \ (a-p)(\beta-\alpha_1)(\beta-\alpha_2)\ldots(\beta-\alpha_n)=0.$

But no one of the factors $(\beta-\alpha_1)$, $(\beta-\alpha_2)\ldots(\beta-\alpha_n)$ is zero.

$$\therefore \ a-p=0. \quad i.e. \ a=p.$$

$\therefore$ from (1), $bx^{n-1}+cx^{n-2}+\ldots=qx^{n-1}+rx^{n-2}+\ldots$ for more than $n-1$ values of x.

$\therefore$ from the above $b=q$, and similarly $c=r$, and so on.

$\therefore$ the theorem is established.

Examples. XXXV. c.

1. Find the condition that $x^2-8ax+p$ may be a perfect square for all values of x.

2. Prove that x^n-1 is not divisible by $x+a$, unless a is equal to ± 1.

3. Prove that $x^{2n+1}+1$ is not divisible by $x+a$, unless a is equal to unity.

4. What number must be added to x^3-5x^2 to make the expression exactly divisible by $x-1$?

5. If $x^2-5px+q^2$ is exactly divisible by $x-p$, prove that $q=\pm 2p$.

6. Find the relation between p, q, and r, if x^3-px^2+qx+r is exactly divisible by $x-p$.

7. If $a+b+c=0$, prove that $a^3+b^3+c^3=3abc$.

8. Prove that $\sum\limits_{xyz}(x-y)^3=3(x-y)(y-z)(z-x)$.

9. Prove that $4px^3-8p^2x^2+3pq^2x+p^2q^2$ is exactly divisible by $x-p$, if $q=\pm p$.

10. Use the remainder theorem to prove that $\sum\limits_{abc}(b+c-2a)^2-9\sum\limits_{abc}(a^2)$ is exactly divisible by $a+b+c$.

11. Solve the equation $(x-a)^2+(y-b)^2=0$. Distinguish between it and the equation $(x-a)(y-b)=0$.

12. Prove that $\sum\limits_{abc}(b+c-2a)^3\equiv 3(b+c-2a)(c+a-2b)(a+b-2c)$.

13. Prove that if $\sum\limits_{abc}(a^2)=\sum\limits_{abc}(bc)$, then $a=b=c$.

14. Find the condition that $ax-p$ may be a factor of ax^2+bx+c.

15. Prove that the equation $(ax-by+2ab)^2+(ax+by)^2=0$ has only one solution, and find that solution.

16. What value of x will make $x^4-10x^3+29x^2-16x-8$ a perfect square?

17. What must be added to $x^4-6x^3+17x^2-32x+12$ in order that it may be a perfect square for all values of x?

18. Prove that $(3a-b)^3+(3b-c)^3+(3c-a)^3=3(3a-b)(3b-c)(3c-a)$ if $a+b+c=0$.

19. Prove that $\frac{p^2}{3}=q=3r^2$, if $x^3+px^2+qx+r^3$ is a perfect cube.

20. Prove that $(x+y+z)^7-x^7-y^7-z^7$ is divisible by $(x+y)(y+z)(z+x)$.

21. Prove that the equation $8x^2-4xy+y^2-4x+1=0$ has only one solution, and find that solution.

22. Prove that $a\left(\frac{b^3-c^3}{b-c}\right)+b\left(\frac{c^3-a^3}{c-a}\right)+c\left(\frac{a^3-b^3}{a-b}\right)=0$, if $a+b+c=0$.

23. Find an integral value of x which will make $(x+1)^2$ greater than $5x-1$ and less than $7x-3$.

24. Prove that if $x=a+d$, $y=b+d$, $z=c+d$, then
$$x^2+y^2+z^2-yz-xz-xy=a^2+b^2+c^2-bc-ca-ab.$$

25. Prove that if $\left(\frac{y}{x}+\frac{b}{a}\right)\left(\frac{z}{x}+\frac{c}{a}\right)=\frac{bc}{a^2}$, then $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=0$.

26. Resolve the following expression into two factors,
$$4x^6-4x^4y^2+12x^3y+x^2y^4-6xy^3+9y^2.$$

27. Find what number must be added to the expression
$$\frac{x^2}{9}+\frac{1}{4x^2}+\frac{2x}{3}+\frac{1}{x} \text{ to make it a perfect square.}$$

28. Simplify the expression
$$(b-c)(x-a)^2+(c-a)(x-b)^2+(a-b)(x-c)^2+(b-c)(c-a)(a-b).$$

29. Simplify the following $\dfrac{x^3(y^2-z^2)+y^3(z^2-x^2)+z^3(x^2-y^2)}{x(y^2-z^2)+y(z^2-x^2)+z(x^2-y^2)}$.

30. If $(b+c)x=a$, $(c+a)y=b$, $(a+b)z=c$, prove that
$$yz+zx+xy+2xyz=1.$$

31. If S_n stands for $x^n+\frac{1}{x^n}$, prove that

(i) $S_1{}^2=S_2+2$. (ii) $S_n{}^2=S_{2n}+2$.
(iii) $S_1S_2=S_3+S_1$. (iv) $S_nS_1=S_{n+1}+S_{n-1}$.

CHAPTER XXXVI.

SURDS.

209. When the root of a quantity cannot be obtained exactly, that root is called a **surd** or **irrational quantity**.

$$\sqrt{2}=1\cdot41421\ldots,\quad \sqrt{7}=2\cdot645\ldots,\quad \sqrt[3]{3}=1\cdot442\ldots$$

are examples of surds.

$\sqrt{9}=3$, and $\sqrt{49}=7$. $\therefore$ $\sqrt{9}$ and $\sqrt{49}$ are *rational* quantities.

By continuing the operation of finding the square root of 2, we can obtain its value to as many decimal places as we please, but its exact value cannot be found. We might express this in another way: no exact quantity multiplied by itself has a product which is 2.

Surds may often be simplified by the use of factors.

Thus $\sqrt{147}=\sqrt{49\,.\,3}=7\sqrt{3}$, and $\sqrt{12}=\sqrt{4\,.\,3}=2\sqrt{3}$.

It must be remembered that $\sqrt{ab} = \sqrt{a}\,.\,\sqrt{b}$, **but $\sqrt{a+b}$ is not equal to $\sqrt{a}+\sqrt{b}$.**

Care must be taken when the quantity is a decimal.

$\sqrt{1{\cdot}69} = 1{\cdot}3$, and is not a surd, but $\sqrt{16{\cdot}9}$ is a surd.

$\sqrt{1{\cdot}44} = 1{\cdot}2$ .... $\sqrt{{\cdot}144}$ ....

Similar or Like Surds are those which are rational multiples of the same surd.

$\sqrt{28} = \sqrt{4.7} = 2\sqrt{7}$, $\sqrt{175} = \sqrt{25.7} = 5\sqrt{7}$, $\sqrt{700} = \sqrt{100.7} = 10\sqrt{7}$; these are *like* surds.

Conversion into an entire surd would be the reverse of this process.

Examples. $2\sqrt{3} = \sqrt{4}\,.\,\sqrt{3} = \sqrt{12}$. $\quad 8\sqrt{6} = \sqrt{64\,.\,6} = \sqrt{384}$.

Expressions can often be simplified by the use of factors.

Example.
$$\begin{aligned}\sqrt{12}+\sqrt{75}-2\sqrt{27} &= \sqrt{4\,.\,3}+\sqrt{25\,.\,3}-2\sqrt{9\,.\,3}\\ &= 2\sqrt{3}+5\sqrt{3}-6\sqrt{3} = \sqrt{3}.\end{aligned}$$

Important. *To find the value of* $\dfrac{6}{\sqrt{2}}$ *correct to* 3 *decimal places.*

We might say that $\dfrac{6}{\sqrt{2}} = \dfrac{6}{1{\cdot}41421}$ approx., and then find the value of the expression by division.

A much shorter method is to **rationalise the denominator** first.

$$\begin{aligned}\frac{6}{\sqrt{2}} = \frac{6\sqrt{2}}{\sqrt{2}\,.\,\sqrt{2}} = \frac{6\sqrt{2}}{2} &= 3\sqrt{2} = 3 \times 1{\cdot}41421\\ &= 4{\cdot}243 \text{ correct to 3 decimal places.}\end{aligned}$$

Examples. XXXVI. a.

(These may be taken orally, if preferred.)

Write down, or read off, as entire surds:

1. $3\sqrt{2}$.	**2.** $5\sqrt{2}$.	**3.** $3\sqrt{5}$.	**4.** $5\sqrt{3}$.
5. $7\sqrt{6}$.	**6.** $2\sqrt{8}$.	**7.** $\dfrac{6}{\sqrt{2}}$.	**8.** $\dfrac{10}{\sqrt{5}}$.
9. $\dfrac{7}{\sqrt{7}}$.	**10.** $\dfrac{9}{\sqrt{3}}$.	**11.** $\dfrac{8}{\sqrt{2}}$.	**12.** $\dfrac{6\sqrt{3}}{\sqrt{2}}$.

Simplify:

13. $\sqrt{12}$. 14. $\sqrt{8}$. 15. $\sqrt{32}$. 16. $\sqrt{75}$.

17. $\sqrt{245}$. 18. $\sqrt{243}$. 19. $\sqrt[3]{81}$. 20. $\sqrt[3]{-81}$

21. $\sqrt[3]{16}$. 22. $\sqrt[3]{32}$. 23. $\sqrt{500}$. 24. $\sqrt{507}$.

25. $\sqrt{2}+\dfrac{2}{\sqrt{2}}$. 26. $2\sqrt{3}+\dfrac{3}{\sqrt{3}}$. 27. $\sqrt{18}+\sqrt{2}$

28. $\sqrt{75}+2\sqrt{3}$. 29. $2\sqrt{5}+\dfrac{10}{\sqrt{5}}$. 30. $\sqrt{8}-\sqrt{2}$.

210. The product of two surdic expressions is found by multiplying each term of one by each term of the other, as in ordinary algebraic multiplication.

Example. $(5\sqrt{3}+2\sqrt{2})\times(3\sqrt{3}-\sqrt{2})$

$$=(5\sqrt{3}+2\sqrt{2})\times 3\sqrt{3}-(5\sqrt{3}+2\sqrt{2})\times\sqrt{2}$$
$$=45+6\sqrt{6}-5\sqrt{6}-4=41+\sqrt{6}.$$

Or

$$\begin{array}{r}
5\sqrt{3}+2\sqrt{2}\\
\underline{3\sqrt{3}-\sqrt{2}}\\
45+6\sqrt{6}\\
\underline{-5\sqrt{6}-4}\\
41+\sqrt{6}
\end{array}$$

Results in surds are only practically useful when expressed as decimals.

The process is much shortened by simplifying the expression first, and by **rationalising the denominator** if the expression is in a fractional form.

We can obtain any required degree of accuracy by taking the root to as many decimal places as we please.

Such cases as $\dfrac{5}{\sqrt{2}}$ we have noticed in Art. 209.

Example 1. To find the value of $\dfrac{1}{3-\sqrt{2}}$ correct to three decimal places.

$$(3-\sqrt{2})\times(3+\sqrt{2})=3^2-(\sqrt{2})^2 \qquad \left[\begin{array}{r}(a+b)(a-b)=a^2-b^2,\\ (\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y})=x-y.\end{array}\right]$$
$$=9-2.$$

$\therefore$ we multiply numerator and denominator by $3+\sqrt{2}$.

Hence
$$\frac{1}{3-\sqrt{2}}=\frac{3+\sqrt{2}}{(3-\sqrt{2})(3+\sqrt{2})}=\frac{3+\sqrt{2}}{9-2}=\frac{4\cdot41421}{7}$$
$$=\cdot631 \text{ correct to 3 decimal places.}$$

Example 2. Find the value of $\dfrac{1}{2+\sqrt{5}-\sqrt{3}}$ correct to 2 decimal places.

(We shall first rationalise the $\sqrt{3}$ in the denominator by multiplying numerator and denominator by $2+\sqrt{5}+\sqrt{3}$.)

$$\frac{1}{2+\sqrt{5}-\sqrt{3}}=\frac{2+\sqrt{5}+\sqrt{3}}{(2+\sqrt{5})^2-(\sqrt{3})^2}=\frac{2+\sqrt{5}+\sqrt{3}}{4+4\sqrt{5}+5-3}=\frac{2+\sqrt{5}+\sqrt{3}}{4\sqrt{5}+6}$$

$$=\frac{2+\sqrt{5}+\sqrt{3}}{2(2\sqrt{5}+3)}=\frac{(2+\sqrt{5}+\sqrt{3})(2\sqrt{5}-3)}{2(2\sqrt{5}+3)(2\sqrt{5}-3)}$$

$$=\frac{4\sqrt{5}+10+2\sqrt{15}-6-3\sqrt{5}-3\sqrt{3}}{2(20-9)}=\frac{4+\sqrt{5}-3\sqrt{3}+2\sqrt{15}}{22}.$$

$$\begin{array}{rr}
 & 4\cdot 0000 \\
\sqrt{5}= & 2\cdot 2361 \\
2\sqrt{15}=2\times 3\cdot 8730= & 7\cdot 7460 \\ \hline
 & 13\cdot 9821 \\
3\sqrt{3}=3\times 1\cdot 73205= & 5\cdot 1962 \\ \hline
2 \,| & 8\cdot 7859 \\
11 \,| & 4\cdot 3930 \\ \hline
 & \cdot 399\ldots
\end{array}$$

$\therefore$ the given fraction $=\cdot 40$ correct to two decimal places.

Examples. XXXVI. b.

Simplify:

1. $\sqrt{12}+2\sqrt{48}+5\sqrt{147}-4\sqrt{3}$.

2. $3\sqrt{125}-2\sqrt{80}+\sqrt{578}$.

3. $3\sqrt{24}-5\sqrt{54}+\sqrt{150}$.

4. $\sqrt{80}+2\sqrt{245}-\sqrt{3125}$.

5. $\sqrt[3]{2187}-2\sqrt[3]{24}$.

6. $\sqrt[3]{108}+10\sqrt[3]{32}+\sqrt[3]{500}$.

7. $7\sqrt{3}-\dfrac{12}{\sqrt{3}}+\sqrt{75}$.

8. $\sqrt{2}(5\sqrt{3}-\sqrt{2})-\sqrt{3}(2\sqrt{2}-\sqrt{3})$.

9. $(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})$.

10. $\left(\dfrac{5}{\sqrt{2}}-\sqrt{2}\right)4\sqrt{2}$.

11. $\left(\dfrac{1}{\sqrt{2}}+\dfrac{1}{\sqrt{8}}\right)\sqrt{2}$.

12. $(\sqrt{7}-2)(\sqrt{7}+2)$.

13. $(\sqrt{3}+\sqrt{2})^2$.

14. $\left(\sqrt{2}+\dfrac{1}{\sqrt{2}}\right)^2$.

15. $(2\sqrt{3}-\sqrt{2})(2\sqrt{3}+\sqrt{2})$.

16. $(\sqrt{5}+1)(2\sqrt{5}-2)$.

17. $(2\sqrt{3}+1)(3\sqrt{3}-1)$.

18. $(3\sqrt{2}-2\sqrt{3})(5\sqrt{2}-\sqrt{3})$.

19. $(5\sqrt{3}-4\sqrt{2})(3\sqrt{3}+2\sqrt{2})$.

20. $(6\sqrt{7}+\sqrt{15})(\sqrt{7}-\sqrt{3})$.

21. $(9\sqrt{2}+5\sqrt{3})(9\sqrt{2}-5\sqrt{3})$.

22. $(3\sqrt{5}+2\sqrt{3})(3\sqrt{5}-2\sqrt{3})$.

Rationalise the denominators of, and express in their simplest forms:

23. $\sqrt{\dfrac{7}{2\cdot1}}$ 24. $\sqrt{\dfrac{7}{0\cdot8}}$ 25. $\sqrt{\dfrac{6}{1\cdot2}}$ 26. $\sqrt{\dfrac{5}{0\cdot4}}$.

27. $\dfrac{1}{\sqrt{2}+1}$. 28. $\dfrac{1}{2-\sqrt{2}}$. 29. $\dfrac{\sqrt{2}-1}{\sqrt{2}+1}$. 30. $\dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}+\sqrt{b}}$.

31. $\dfrac{1}{3+2\sqrt{2}}$. 32. $\dfrac{4}{\sqrt{5}+1}$. 33. $\dfrac{3}{\sqrt{5}+\sqrt{2}}$. 34. $\dfrac{2+4\sqrt{7}}{2\sqrt{7}+1}$.

35. $\dfrac{5+2\sqrt{6}}{6-2\sqrt{6}}$. 36. $\dfrac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$.

[$\sqrt{2}=1\cdot41421$, $\sqrt{3}=1\cdot73205$, $\sqrt{5}=2\cdot2361$, $\sqrt{6}=2\cdot4495$, $\sqrt{7}=2\cdot6458$. The above values may be used in the following examples.]

Calculate to two decimal places the value of

37. $\dfrac{1}{\sqrt{5}-1}$. 38. $\dfrac{4}{3-2\sqrt{2}}$. 39. $\dfrac{7\sqrt{2}+3}{7\sqrt{2}-3}$. 40. $\dfrac{4\sqrt{7}+3\sqrt{2}}{5\sqrt{2}+2\sqrt{7}}$.

41. $3-\dfrac{2}{\sqrt{6}}$. 42. $(\sqrt{3}-\sqrt{2})^2$. 43. $\dfrac{\sqrt{3}-1}{\sqrt{2}-1}$. 44. $\dfrac{3}{\sqrt{7}-\sqrt{3}}$.

45. $\dfrac{3\sqrt{3}-1}{3\sqrt{2}-1}$. 46. $\dfrac{57}{5\sqrt{3}-3\sqrt{2}}$. 47. $1\div\dfrac{\sqrt{5}-1}{4}$. 48. $\dfrac{1}{\sqrt{3}+\sqrt{2}-\sqrt{5}}$.

49. $\dfrac{12}{2+\sqrt{3}-\sqrt{7}}$. 50. $\dfrac{1}{\sqrt{5}+\sqrt{3}+2\sqrt{2}}$. 51. $\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}+\sqrt{2}-1}$.

211. *If $a+\sqrt{b}=c+\sqrt{d}$, where a, c are rational and $\sqrt{b}$, $\sqrt{d}$ are surds, then $a=c$ and $b=d$.*

For if not, let
$$a=c+x,$$
$$\therefore\ x+\sqrt{b}=\sqrt{d},$$
$$\therefore\ \text{by squaring, } x^2+2x\sqrt{b}+b=d;$$
$$\therefore\ 2x\sqrt{b}=d-b-x^2,$$
i.e. a surd = a rational quantity, which is impossible.
$$\therefore\ a=c \text{ and } \sqrt{b}=\sqrt{d}.$$

212. *The square of the sum of two surds = a rational quantity + a surd,* *e.g.* $(\sqrt{5}+\sqrt{3})^2=8+2\sqrt{15}$.

Consequently the sq. rt. of $a+\sqrt{b}$ may sometimes be found in the form $\sqrt{x}+\sqrt{y}$.

213. If $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$ we have, by squaring,

$$a+\sqrt{b}=x+y+2\sqrt{xy}.$$

Equate rational to rational and surd to surd.

Then $$a=x+y, \quad \sqrt{b}=2\sqrt{xy};$$

$$\therefore \; a-\sqrt{b}=x+y-2\sqrt{xy}.$$

Thus if $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$, then $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$.

Example. Find $\sqrt{10+2\sqrt{21}}$.

Let $\sqrt{10+2\sqrt{21}}=\sqrt{x}+\sqrt{y}$.

Then $\sqrt{10-2\sqrt{21}}=\sqrt{x}-\sqrt{y}$.

$\therefore$ by multiplication,

$$\sqrt{100-84}=x-y;$$

$$\therefore \; 4=x-y.$$

By squaring we get

$$10+2\sqrt{21}=x+y+2\sqrt{xy}.$$

By equating rational parts,

$$x+y=10.$$

But $$x-y=4;$$

$$\therefore \; x=7, \quad y=3.$$

$$\therefore \; \sqrt{10+2\sqrt{21}}=\sqrt{7}+\sqrt{3}.$$

Square roots of surdic expressions can sometimes be determined by inspection, by remembering that

$$(a\pm b)^2=a^2+b^2\pm 2ab, \text{ and } (\sqrt{x}\pm\sqrt{y})^2=x+y\pm 2\sqrt{xy}.$$

Thus $$4-2\sqrt{3}=3+1-2\sqrt{3}.$$

$$\therefore \; \sqrt{4-2\sqrt{3}}=\sqrt{3}-1.$$

Also $$10+2\sqrt{21}=7+3+2\sqrt{7\,.\,3}.$$

$$\therefore \; \sqrt{10+2\sqrt{21}}=\sqrt{7}+\sqrt{3}.$$

Examples. XXXVI. c.

Find the sq. root (evaluating results to 2 decimal places) of

1. $4+2\sqrt{3}$. 2. $7+2\sqrt{6}$. 3. $12-6\sqrt{3}$. 4. $11+6\sqrt{2}$.
5. $30+4\sqrt{14}$. 6. $17-12\sqrt{2}$. 7. $12+2\sqrt{35}$. 8. $32-8\sqrt{15}$.
9. $\frac{9}{4}-\sqrt{5}$. 10. $27+4\sqrt{35}$. 11. $101-28\sqrt{13}$.
12. $\frac{73}{36}+\frac{1}{3}\sqrt{2}$. 13. $4\sqrt{2}-2\sqrt{6}$. 14. $33-18\sqrt{2}$.

Find the 4th roots (leaving your results in surdic form) of

15. $49+20\sqrt{6}$. 16. $17+12\sqrt{2}$.
17. $56-24\sqrt{5}$. 18. $284-48\sqrt{35}$.
19. Find to four decimal places the value of

$$\frac{1}{\sqrt{\{12-\sqrt{(56-24\sqrt{5})}\}}}.$$

20. Simplify $\left(\frac{10+9\sqrt{5}}{9+2\sqrt{5}}\right)^2$.

21. Calculate to 3 decimal places $\frac{(3+\sqrt{5})^2}{3}-\frac{(2+\sqrt{10})^2}{\sqrt{8}}$.

22. Simplify $\frac{\sqrt{3}-1}{\sqrt{5}-2}\times\frac{\sqrt{10}-2\sqrt{2}}{3-\sqrt{3}}$, and find its value to 3 places of decimals.

23. Find the value of $\left(\frac{x+a}{x-a}\right)^2-\frac{x}{3a}$ when $x=a(1+2\sqrt{3})$.

24. Reduce $\sqrt{6-\sqrt{17-12\sqrt{2}}}$ to its simplest form.

25. Simplify $\frac{\sqrt{3}}{\sqrt{5}+\sqrt{2}}-\frac{\sqrt{3}}{\sqrt{500}-\sqrt{200}}$ and $\frac{5}{\sqrt{15}+\sqrt{6}}-\frac{1}{\sqrt{60}-\sqrt{24}}$.

Find the value of their product to 3 decimal places.

26. Simplify $\frac{\sqrt{245}+\sqrt{75}}{\sqrt{5}-\sqrt{3}}+\frac{\sqrt{245}-\sqrt{75}}{\sqrt{5}+\sqrt{3}}$.

27. If $a(b-c)^2=c(b+c)^2$, find the value of $\frac{c}{b}\cdot\frac{\sqrt{a}+\sqrt{c}}{\sqrt{a}-\sqrt{c}}$.

28. Simplify $\frac{2(1+\sqrt{3})}{1-\sqrt{2}+\sqrt{3}}$.   29. Simplify $\sqrt{1+\sqrt{1-a^2}}+\sqrt{1-\sqrt{1-a^2}}$.

Find the product of

30. $\sqrt{2+\sqrt{3}}$, $\sqrt{2+\sqrt{2+\sqrt{3}}}$, $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{3}}}}$, $\sqrt{2-\sqrt{2+\sqrt{2+\sqrt{3}}}}$, the positive value of the root being taken in each case.

CHAPTER XXXVII.

INDICES.

214. "What is the meaning of a^5?" A very common answer is "a multiplied by itself 5 times." Of course this is wrong. There are 5 factors, but only 4 multiplications. a^5 is the product of 5 factors, each factor being a.

$$a^n = a\,.\,a\,.\,a\ldots n \text{ factors},\quad a^p = a\,.\,a\,.\,a\ldots p \text{ factors}.$$

$$\therefore\; a^n\,.\,a^p = a\,.\,a\,.\,a\ldots n \text{ factors}\times a\,.\,a\,.\,a\ldots p \text{ factors}$$

$$= a\,.\,a\,.\,a\ldots \overline{n+p} \text{ factors} = a^{n+p}. \quad\ldots\ldots\ldots\ldots\ldots(1)$$

$$\frac{a^n}{a^p}=\frac{a\,.\,a\,.\,a\ldots n \text{ factors}}{a\,.\,a\,.\,a\ldots p \text{ factors}}.$$

If there are more factors in the numerator than in the denominator, cancelling will leave $n-p$ factors in the numerator; but if there are more in the denominator, cancelling will leave $p-n$ factors in the denominator.

Thus $$\frac{a^n}{a^p} = a^{n-p} \text{ if } n>p \qquad\qquad (2)$$
$$= \frac{1}{a^{p-n}} \text{ if } n<p.$$

215. These results depend upon the definition "a^n is the product of n factors a"; and this definition evidently requires that n should be a positive integer. If we wish to employ fractional or negative indices, it is necessary to give a definition which is applicable to them.

DEF. **Fractional and negative indices are defined as being such that they obey this law:—To multiply powers of a quantity add the indices.**

216. *To find the meaning of* $a^{\frac{1}{n}}$.

Following the above law,

$$a^{\frac{1}{2}} \times a^{\frac{1}{2}} = a^{\frac{1}{2}+\frac{1}{2}} = a^1 = a.$$

But $$\sqrt{a} \times \sqrt{a} = a.$$

We therefore denote $\sqrt{a}$ by $a^{\frac{1}{2}}$.

In the same way, $$a^{\frac{1}{3}} \times a^{\frac{1}{3}} \times a^{\frac{1}{3}} = a^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = a.$$

But $$\sqrt[3]{a} \times \sqrt[3]{a} \times \sqrt[3]{a} = a.$$

We therefore denote $\sqrt[3]{a}$ by $a^{\frac{1}{3}}$.

Similarly, $a^{\frac{1}{n}}$ when raised to the n^{th} power becomes a.

We therefore denote $\sqrt[n]{a}$ by $a^{\frac{1}{n}}$.

Example. $$8^{\frac{1}{3}} = \sqrt[3]{8} = 2.$$

217. *To find the meaning of* a^0.

$$a^0 = a^n \div a^n \quad \text{(following the above law)}$$

$$\therefore = \frac{a^n}{a^n} = 1,$$

i.e. **$a^0 = 1$, for all values of a.**

Examples. $2^0 = 1, \quad (-2)^0 = 1, \quad (\sqrt{3})^0 = 1.$

$$3^{\frac{2}{3}} \times 3^{-\frac{2}{3}} = 3^{\frac{2}{3}-\frac{2}{3}} = 3^0 = 1.$$

218. *To find the meaning of a^{-n}, n being a positive integer.*

$$a^{-n} = \frac{a^{-n} \times a^n}{a^n} = \frac{a^{n-n}}{a^n} = \frac{a^0}{a^n} = \frac{1}{a^n}.$$

$$\therefore \; \mathbf{a^{-n} = \frac{1}{a^n}} \textbf{ for all values of a.}$$

Examples. $2^{-3} = \frac{1}{2^3} = \frac{1}{8}. \qquad 3^{-2} = \frac{1}{3^2} = \frac{1}{9}.$

$$\left(\frac{1}{2}\right)^{-4} = 1 \div \left(\frac{1}{2}\right)^4 = 1 \div \frac{1}{2^4} = 2^4 = 16.$$

219. Following the same law, m, n, p, q being positive integers,

$$\mathbf{a}^{\frac{m}{n}} \times \mathbf{a}^{\frac{p}{q}} = \mathbf{a}^{\frac{m}{n}+\frac{p}{q}}.$$

Also
$$a^{\frac{m}{n}-\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{m}{n}-\frac{p}{q}+\frac{p}{q}} = a$$

$$\therefore \; \frac{\mathbf{a}^{\frac{m}{n}}}{\mathbf{a}^{\frac{p}{q}}} = \mathbf{a}^{\frac{m}{n}-\frac{p}{q}}.$$

Examples.

$$a^{\frac{3}{4}} \times a^{\frac{1}{4}} = a^{\frac{3}{4}+\frac{1}{4}} = a.$$

$$a^{\frac{3}{4}} \div a^{\frac{1}{4}} = a^{\frac{3}{4}-\frac{1}{4}} = a^{\frac{1}{2}} = \sqrt{a}.$$

$$3^{\frac{3}{2}} \times 3^{\frac{1}{2}} = 3^{\frac{3}{2}+\frac{1}{2}} = 3^2 = 9.$$

220. *To prove that $(a^m)^n = a^{mn}$ for all values of m and n.*

(1) When n is a positive integer (m having any value),

$$(a^m)^n = a^m \times a^m \times a^m \ldots n \text{ factors}$$
$$= a^{m+m+m \ldots \text{ to } n \text{ terms}} = a^{mn} \quad \ldots\ldots\ldots\ldots\ldots\ldots(a)$$

(2) When n is a positive fraction (m having any value),

take $n = \frac{p}{q}$, p and q being positive integers.

$$x = \sqrt[q]{x^q}; \quad \therefore \text{ writing } (a^m)^{\frac{p}{q}} \text{ instead of } x,$$

we have $(a^m)^{\frac{p}{q}} = \sqrt[q]{\left\{(a^m)^{\frac{p}{q}}\right\}^q} = \sqrt[q]{(a^m)^p}$, by (a) above,

$$= \sqrt[q]{a^{mp}} = a^{\frac{mp}{q}}.$$

(3) Lastly, when n is negative, fractional or integral (m having any value),

take $n = -s$, s being positive.

$$(a^m)^n = (a^m)^{-s} = \frac{1}{(a^m)^s} = \frac{1}{a^{ms}} = a^{-ms} = a^{mn}.$$

Examples.

$$(a^3)^{\frac{2}{3}} = a^{3\times\frac{2}{3}} = a^2.$$
$$(a^5)^{-\frac{3}{5}} = a^{-3} = \frac{1}{a^3}.$$
$$(2^6)^{-\frac{1}{3}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}.$$

221. *To prove that $(ab)^n = a^n b^n$ for all values of n.*

(1) When n is a positive integer,

$(ab)^n = ab \times ab \times ab \ldots$ (the product containing n factors ab)

$= a^n b^n$. .... (1)

(2) When n is a positive fraction,

take $n = \frac{p}{q}$, p and q being positive integers.

$$\left(a^{\frac{p}{q}} . b^{\frac{p}{q}}\right)^q = a^p . b^p, \text{ from (1)},$$
$$= (ab)^p = \left\{(ab)^{\frac{p}{q}}\right\}^q;$$

$\therefore$ taking the q^{th} root of each side,

we have $$a^{\frac{p}{q}} . b^{\frac{p}{q}} = (ab)^{\frac{p}{q}}.$$

(3) Lastly, if n is negative,

let $n = -s$, where s is positive.

$$(ab)^n = (ab)^{-s} = \frac{1}{(ab)^s} = \frac{1}{a^s . b^s} = a^{-s} . b^{-s}$$
$$= a^n . b^n.$$

Examples. $$\left(a^{\frac{5}{4}} b^{\frac{5}{2}}\right)^4 = a^5 b^{10}.$$

$$\left(8^{\frac{1}{2}} . 3^{\frac{3}{2}}\right)^{\frac{1}{3}} = \left(2^{\frac{3}{2}} . 3^{\frac{3}{2}}\right)^{\frac{1}{3}} = 2^{\frac{1}{2}} . 3^{\frac{1}{2}} = \sqrt{6}.$$

Simplify the expression $$\frac{2^n . 8^{n+2} . 4^{-3n}}{2^{-2n}}.$$

(Express it in powers of 2.)

$$\text{The expression} = 2^n . (2^3)^{n+2} . (2^2)^{-3n} . 2^{2n}$$
$$= 2^n . 2^{3n+6} . 2^{-6n} . 2^{2n}$$
$$= 2^{n+3n+6-6n+2n} = 2^6 = 64.$$

222. In multiplication and division of compound expressions, we work, both as to method and arrangement, as if the indices were positive integers.

Example 1. Multiply $\sqrt[3]{a^2}-2\sqrt{b^{-1}}$ by $a^{\frac{1}{3}}+b^{-\frac{1}{4}}$.

$$\begin{array}{l}
a^{\frac{2}{3}}-2b^{-\frac{1}{2}} \\
\underline{a^{\frac{1}{3}}+\ b^{-\frac{1}{4}}\qquad\qquad} \\
a\ -2a^{\frac{1}{3}}b^{-\frac{1}{2}} \\
\underline{\quad +a^{\frac{2}{3}}b^{-\frac{1}{4}}-2b^{-\frac{3}{4}}\qquad\qquad} \\
a+a^{\frac{2}{3}}b^{-\frac{1}{4}}-2a^{\frac{1}{3}}b^{-\frac{1}{2}}-2b^{-\frac{3}{4}}
\end{array}$$

Example 2. Multiply $x^{\frac{1}{2}}+x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$ by $x^{\frac{1}{2}}-x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$.

$$\left\{\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)+x^{\frac{1}{4}}y^{\frac{1}{4}}\right\}\cdot\left\{\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)-x^{\frac{1}{4}}y^{\frac{1}{4}}\right\}$$
$$=\left(x^{\frac{1}{2}}+y^{\frac{1}{2}}\right)^2-\left(x^{\frac{1}{4}}y^{\frac{1}{4}}\right)^2=x+2x^{\frac{1}{2}}y^{\frac{1}{2}}+y-x^{\frac{1}{2}}y^{\frac{1}{2}}$$
$$=x+x^{\frac{1}{2}}y^{\frac{1}{2}}+y.$$

Example 3. Divide $a^2+4a^{\frac{3}{2}}b^{-1}+6ab^{-2}+4a^{\frac{1}{2}}b^{-3}+b^{-4}$ by $a^{\frac{1}{2}}+b^{-1}$.

$$\begin{array}{l}
a^{\frac{1}{2}}+b^{-1}\big)\,a^2+4a^{\frac{3}{2}}b^{-1}+6ab^{-2}+4a^{\frac{1}{2}}b^{-3}+b^{-4}\big(\,a^{\frac{3}{2}}+3ab^{-1}+3a^{\frac{1}{2}}b^{-2}+b^{-3} \\
\qquad\qquad\underline{a^2+\ a^{\frac{3}{2}}b^{-1}} \\
\qquad\qquad\qquad 3a^{\frac{3}{2}}b^{-1}+6ab^{-2} \\
\qquad\qquad\qquad \underline{3a^{\frac{3}{2}}b^{-1}+3ab^{-2}} \\
\qquad\qquad\qquad\qquad\quad 3ab^{-2}+4a^{\frac{1}{2}}b^{-3} \\
\qquad\qquad\qquad\qquad\quad \underline{3ab^{-2}+3a^{\frac{1}{2}}b^{-3}} \\
\qquad\qquad\qquad\qquad\qquad\qquad a^{\frac{1}{2}}b^{-3}+b^{-4} \\
\qquad\qquad\qquad\qquad\qquad\qquad \underline{a^{\frac{1}{2}}b^{-3}+b^{-4}}
\end{array}$$

Examples. XXXVII. a.

(These examples may be taken orally.)

Write down, or read off, in their simplest forms:

1. $16^{\frac{1}{2}}$. 2. 2^{-3}. 3. $2^{2}\times 2^{3}$. 4. $8^{\frac{1}{3}}$.

5. $27^{\frac{1}{3}}$. 6. $4^{-\frac{1}{2}}$. 7. $3^{\frac{1}{2}}\times 3^{-\frac{1}{2}}$. 8. $a^2\div a^{\frac{3}{2}}$.

9. $a^2\div a^{-\frac{1}{2}}$. 10. $a^{\frac{3}{4}}\div a^{\frac{1}{4}}$. 11. $2x^{\frac{5}{4}}\div x^{\frac{1}{4}}$. 12. $(a^2)^{-\frac{1}{2}}$.

13. $\dfrac{1}{2^{-2}}$. 14. $\left(\dfrac{1}{3^2}\right)^{-1}$. 15. $4^{\frac{3}{2}}$. 16. $25^{-\frac{1}{2}}$.

Write down, or read off, in their simplest forms:

17. $64^{\frac{1}{6}}$. 18. $(6^3)^{\frac{2}{3}}$. 19. 2×2^{-3}. 20. $\frac{1}{3^{-4}}$.

21. $\frac{1}{4^{\frac{1}{2}}}$. 22. $\frac{1}{8^{-\frac{1}{3}}}$. 23. $16^{\frac{5}{4}}$. 24. $16^{-\frac{5}{2}}$.

25. $81^{\frac{3}{4}}$. 26. $3^{\frac{1}{3}} \times 9^{\frac{1}{3}}$. 27. $16^{\frac{3}{4}} \times 16^{-\frac{1}{4}}$. 28. $\left(2^{\frac{3}{2}}\right)^{-\frac{2}{3}}$.

29. $125^{\frac{2}{3}}$. 30. $125^{-\frac{2}{3}}$. 31. $3^{n-1} \times 3^{1-n}$. 32. $\frac{5^{\frac{1}{2}}}{5^{-}}$

33. $\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)^2$. 34. $\left(a^{\frac{1}{2}}+b^{\frac{1}{2}}\right)\left(a^{\frac{1}{2}}-b^{\frac{1}{2}}\right)$. 35. $(x+x^{-1})^2$. 36. $x^{a-b} \times x^{a+b}$.

37. $(x^{a+b})^{a-b}$. 38. $(e^x+e^{-x})^2$. 39. $\left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right)^2$. 40. $\left(2x-\frac{x^{-1}}{2}\right)^2$.

Examples. XXXVII. b.

Express as simply as possible with indices, without denominators:

1. $\frac{1}{\sqrt[3]{a^{-1}}}$.
2. $\sqrt[5]{a^2 b}$.
3. $\frac{3\sqrt{b}}{\sqrt[3]{c}}$.
4. $\frac{a^4}{\sqrt[3]{a^2}}$; $\frac{\sqrt[4]{y}}{\sqrt{xy}}$; $\sqrt{2a^{-3}}$.
5. Simplify $8^{\frac{2}{3}}$ and $25^{-\frac{1}{2}}$.
6. Simplify $27^{-\frac{2}{3}}$ and $49^{\frac{3}{2}}$.
7. Express with positive indices $a^{-1}bc+a^{-2}b^{-1}c+ab^{-1}c^{-1}$.

Simplify

8. $16^{\frac{3}{4}}$. 9. $256^{-\frac{3}{8}}$. 10. $289^{-\frac{1}{2}}$. 11. $32^{-\frac{3}{5}}$.

12. $729^{-\frac{4}{3}}$. 13. $625^{\frac{3}{4}}$. 14. $\frac{1024^{\frac{1}{10}}}{125^{-\frac{2}{3}}}$. 15. $\frac{1}{343^{-\frac{2}{3}}}$.

16. $(a^{\frac{1}{6}}x^{-2})^{-3}$. 17. $(\frac{1}{16}a^{12}b^{-8})^{\frac{1}{4}}$. 18. $(27b^{-6}c^3)^{\frac{1}{3}}$. 19. $(2x^{-1}\sqrt[3]{y^2})^{-6}$.

20. $\left(\frac{32a^{-5}}{b^{-10}}\right)^{\frac{1}{5}}$. 21. $\frac{a^{-1}+b^{-1}}{a^{-2}-b^{-2}}$. 22. $\frac{a^{\frac{3}{2}}+b^{\frac{3}{2}}}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$.

23. $\left(1024^{-\frac{1}{5}}\right)^{\frac{1}{2}}$. 24. $\frac{2^n . 4^{n+1}}{8^{n-2}}$. 25. $\frac{5^{-n} . 25^{2n-2}}{5^{3n-2} . 10^{-1}}$.

Multiply

26. $2y+3x^{\frac{1}{4}}y^{\frac{1}{2}}-x^{\frac{1}{2}}$ by $7x^{\frac{1}{4}}+5y^{\frac{1}{2}}$. 27. $\sqrt{x^3}+1+\frac{1}{\sqrt{x^3}}$ by $\sqrt{x^3}-1+\frac{1}{\sqrt{x^3}}$.

28. $a^{-1}+a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$ by $a^{-1}-a^{-\frac{1}{2}}b^{-\frac{1}{2}}+b^{-1}$.

29. $a^{\frac{5}{2}}+a^2b^{\frac{1}{3}}+a^{\frac{3}{2}}b^{\frac{2}{3}}+ab+a^{\frac{1}{2}}b^{\frac{4}{3}}+b^{\frac{5}{3}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{3}}$.

30. $a+2b^{\frac{1}{4}}+3c^{\frac{1}{2}}$ by $a+2b^{\frac{1}{4}}-3c^{\frac{1}{2}}$. 31. $x^{\frac{3}{4}}+x^{\frac{1}{2}}y+x^{\frac{1}{4}}y^2+y^3$ by $x^{\frac{1}{4}}-y$.

32. $a^2+3ab^{-1}+4c^{-2}$ by $a^2-3ab^{-1}-4c^{-2}$.

Divide

33. $8x^3 - 27y^{-3}$ by $2x - 3y^{-1}$. **34.** $x^{-3} - 64y^2$ by $x^{-\frac{1}{2}} + 2y^{\frac{1}{3}}$.

35. $a - 2a^{\frac{1}{2}}b^{\frac{1}{2}} + b$ by $a^{\frac{1}{2}} - 2a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}}$.

36. Simplify $\left\{\sqrt[3]{4} \times \dfrac{1}{\sqrt[6]{8}} \times \sqrt[12]{2^{-1}}\right\}^4$.

37. Divide $a^3 - b^{-3}$ by $a^{\frac{1}{2}} - b^{-\frac{1}{2}}$.

38. Give the product of $a^{\frac{1}{3}} + a^{\frac{1}{6}}b^{\frac{1}{6}} + b^{\frac{1}{3}}$ and $a^{\frac{1}{3}} - a^{\frac{1}{6}}b^{\frac{1}{6}} + b^{\frac{1}{3}}$.

39. Factorise $\frac{5}{4}\left(a^{\frac{3}{2}} + a^{-\frac{3}{2}}\right) + \frac{1}{4}\left(a^{\frac{1}{2}} + a^{-\frac{1}{2}}\right)$.

40. Square $a^{\frac{1}{2}} - 2a^{\frac{1}{4}}b^{\frac{1}{2}} + b$, and divide the result by $(a^{\frac{1}{4}} - b^{\frac{1}{2}})^3$.

CHAPTER XXXVIII.

RATIO, PROPORTION AND VARIATION.

RATIO.

223. If two quantities are of the same kind, they have a **ratio**; and the ratio of the 1st to the 2nd is the quotient obtained by dividing the 1st by the 2nd, whether that quotient be integral or fractional.

The ratio of a to b is expressed as $a : b$ or $\frac{a}{b}$.

a, b are respectively called the 1st and 2nd *terms* or *members* of the ratio, or the **antecedent** and **consequent**.

If the antecedent = the consequent, the ratio is a **ratio of equality**, and is equal to unity.

If the antecedent is the greater, the ratio is called a **ratio of greater inequality**, *i.e.* an improper fraction.

If the antecedent is the less, the ratio is a **ratio of less inequality**, *i.e.* a proper fraction.

224. *A ratio of greater inequality is diminished by adding the same positive quantity to both its members.*

Let $\frac{a}{b}$ be a ratio of greater inequality (*i.e.* $a > b$).

Let $\frac{a+x}{b+x}$ be the new ratio.

$$\frac{a}{b}-\frac{a+x}{b+x}=\frac{ab+ax-ab-bx}{b(b+x)}=\frac{(a-b)x}{b(b+x)}$$

$$= \text{a positive quantity, for } a>b,$$

i.e. the original − the new ratio = a positive quantity;

$\therefore$ the new ratio < the original ratio.

Similarly it may be proved that *a ratio of less inequality is increased by adding the same positive quantity to both its members.*

The proof of this should be written out as an exercise.

The two statements may be combined in one, viz.: *A ratio is made nearer to unity by adding the same positive quantity to both its members.*

225. Ratios are **compounded** by being multiplied together.

The **duplicate** ratio of a to b is $a^2:b^2$.

The **sub-duplicate** ratio of a to b is $a^{\frac{1}{2}}:b^{\frac{1}{2}}$.

226. Many properties of ratios are easily proved by taking some single letter k to represent a ratio, or to represent each of several equal ratios.

Example. To prove that, if $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, each of these ratios $=\frac{la+mc+ne}{lb+md+nf}$.

Let $\frac{a}{b}=k$. Then $\frac{c}{d}=k$, and $\frac{e}{f}=k$;

$\therefore\ a=bk, \qquad c=dk, \qquad e=fk.$

$$\frac{la+mc+ne}{lb+md+nf}=\frac{lbk+mdk+nfk}{lb+md+nf}=\frac{(lb+md+nf)k}{lb+md+nf}=k=\frac{a}{b}.$$

As a simple case take $\frac{a+c+e}{b+d+f}$. This $=\frac{bk+dk+fk}{b+d+f}=\frac{(b+d+f)k}{b+d+f}=k=\frac{a}{b}$.

From this we see that **if a number of ratios are equal, a new ratio equal to each of them can be formed by adding their antecedents for a new antecedent and adding their consequents for a new consequent.**

NOTE.—A ratio may sometimes be simplified by the use of this Article for purposes of approximation or checking.

Thus $\frac{4526}{1007}=$ (approximately) $\frac{4494\cdot5}{1000}=4\cdot4945$.

The fuller working is

$$\frac{4526}{1007}=4\cdot5 \text{ roughly}=\frac{31\cdot5}{7}=\frac{4526-31\cdot5}{1007-7}=\frac{4494\cdot5}{1000}=4\cdot4945.$$

227. If $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$ are unequal, the ratio $\frac{a+c+e}{b+d+f}$ lies in magnitude between the greatest and least of these ratios.

Suppose $$\frac{a}{b} > \frac{c}{d} > \frac{e}{f}.$$

Let $$\frac{a}{b} = k. \quad \text{Then } \frac{c}{d} < k \text{ and } \frac{e}{f} < k;$$

$$\therefore\ a = bk, \qquad c < dk, \qquad e < fk;$$

$$\therefore\ a+c+e < (b+d+f)k; \quad \therefore\ \frac{a+c+e}{b+d+f} < k.$$

Let $\frac{e}{f} = k'$. Then in a similar way $\frac{a+c+e}{b+d+f} > k'$.

Thus $\frac{a+c+e}{b+d+f}$ lies between the greatest and least of the ratios $\frac{a}{b}, \frac{c}{d}, \frac{e}{f}$.

Example. Find a ratio intermediate between $\frac{7}{8}$ and $\frac{15}{16}$.

By what has been proved we see that such a ratio can be found by adding the numerators and adding the denominators. Result $\frac{22}{24}$, *i.e.* $\frac{11}{12}$.

228. *If* $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, *then each of these ratios is equal to*

$$\left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}\right)^{\frac{1}{n}},$$

p, q, r, n being any quantities whatever.

Let $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = k,$$

so that $$a = bk, \quad c = dk, \quad e = fk,$$

and $$a^n = b^n k^n, \quad c^n = d^n k^n, \quad e^n = f^n k^n.$$

$$\therefore\ pa^n = pb^n k^n, \quad qc^n = qd^n k^n, \quad re^n = rf^n k^n.$$

$\therefore$ by addition,

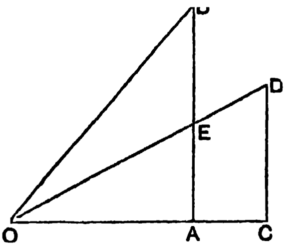
$$pa^n + qc^n + re^n = k^n(pb^n + qd^n + rf^n).$$

$\therefore\ \frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n} = k^n$, and taking the n^{th} root of each side,

$$\left(\frac{pa^n + qc^n + re^n}{pb^n + qd^n + rf^n}\right)^{\frac{1}{n}} = k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}. \qquad \text{Q.E.D.}$$

Graphic Representation of Ratio.

229. Take an abscissa OA to represent the consequent of the ratio, and an ordinate AB to represent the antecedent on the same scale. The magnitude of the angle AOB enables us to estimate whether the ratio is greater or less than another ratio represented in the same manner.



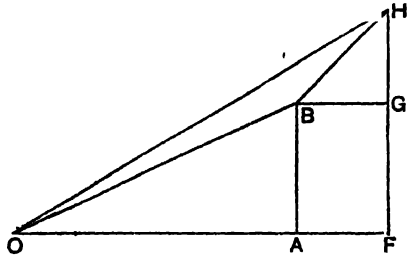
Suppose $\frac{DC}{OC}$ to be the 2nd ratio; and let OD meet AB at E.

By similar $\triangle$'EAO, DCO, $\frac{DC}{OC}=\frac{EA}{OA}$.

$$\therefore \frac{EA}{OA} \text{ also represents the 2nd ratio.}$$

Thus we can compare the ratios by means of the lengths of AB and AE.

A ratio of less inequality is increased by adding the same quantity to both its terms.



As before, take an abscissa OA and ordinate AB, so that $\frac{AB}{OA}$ represents a ratio.

If OA be produced to F, and an ordinate FGH be drawn, where

$FG = AB$ and $GH = AF$, the ratio $\frac{AB}{OA}$ has been altered by adding the same quantity HG or AF to both its terms.

The new ratio $\frac{HF}{OF} >$ the old ratio, if OH cuts AB above B; *i.e.* if the $\angle HBG >$ the $\angle BOA$.

But $\angle HBG = 45°$;

$\therefore$ the new ratio $>$ the old, if $\angle BOA < 45°$ *i.e.* if the ratio $\frac{AB}{OA}$ is one of less inequality.

NOTE. It is convenient to have a name to denote this graphic representation of a ratio. In trigonometry the ratio $AB : OA$ is called the *tangent* of the angle AOB. In many mathematical works this ratio is denoted by the *slope* of the line OB.

230. *Thermometric scales.* In a Fahrenheit thermometer the freezing point is marked 32° and boiling point 212°; in the Centigrade thermometer these are marked 0° and 100° respectively. If a certain temperature be indicated on the Fahrenheit scale by F degrees, and on the Centigrade scale by C degrees, we can compare these by noticing that the distances of the given temperature and the boiling temperature from the freezing point must have the same ratio in whichever scale they are expressed.

Thus

$$\frac{F-32}{212-32} = \frac{\text{distance of the given temperature from freezing point}}{\text{distance between boiling and freezing points}}$$

$$= \frac{C-0}{100-0},$$

$$\text{i.e.} \quad \frac{F-32}{180} = \frac{C}{100}; \qquad \therefore \frac{F-32}{9} = \frac{C}{5}.$$

For the graph of this equation see Article 80, p. 123.

Example 1. If $\frac{7x-3y}{5x+4y} = \frac{29}{33}$, find the ratio $x : y$.

$$33(7x-3y) = 29(5x+4y);$$

$$\therefore 231x - 99y = 145x + 116y;$$

$$\therefore 86x = 215y;$$

$$\therefore x = \frac{215y}{86};$$

$$\therefore \frac{x}{y} = \frac{215}{86} = \frac{5}{2}.$$

Example 2. If $2x^2-5xy+2y^2=0$, find the ratio $x:y$.

$$2\left(\frac{x}{y}\right)^2-5\frac{x}{y}+2=0;$$

$\therefore$ by solving this quadratic equation we get

$$\frac{x}{y}=2 \text{ or } \frac{1}{2}.$$

Example 3. When a straight line is divided in extreme and mean ratio, what are approximately the ratios of the parts to the whole?

Let the whole measure a units, the two parts x and $a-x$ units.

By hypothesis $\quad x^2=a(a-x);$

$$x^2+ax=a^2$$

$\therefore \quad x=a\cdot\frac{\sqrt{5}-1}{2}$, rejecting the negative solution.

$\therefore$ the ratio $\frac{x}{a}=\frac{\sqrt{5}-1}{2}=\cdot 618$ approximately.

Example 4. If $\frac{x}{y}=3$ and $\frac{a}{b}=\frac{2}{5}$, find the value of $\frac{12ax-by}{2ax+3by}$.

$$\frac{12ax-by}{2ax+3by}=\frac{12ax-by}{by}:\frac{2ax+3by}{by}$$

$$=\frac{12\frac{a}{b}\cdot\frac{x}{y}-1}{2\frac{a}{b}\cdot\frac{x}{y}+3}=\frac{12\cdot\frac{2}{5}\cdot 3-1}{2\cdot\frac{2}{5}\cdot 3+3}$$

$$=\frac{\frac{72}{5}-1}{\frac{12}{5}+3}=\frac{72-5}{12+15}=\frac{67}{27}.$$

231. Cross Multiplication. *From the equations $a_1x+b_1y+c_1z=0$, $a_2x+b_2y+c_2z=0$, find the ratios $\frac{x}{z}$ and $\frac{y}{z}$.*

By multiplying the first by b_2 and the second by b_1, we get

$$a_1b_2x+b_1b_2y+b_2c_1z=0,$$

$$a_2b_1x+b_1b_2y+b_1c_2z=0;$$

$\therefore$ by subtraction, $x(a_1b_2-a_2b_1)+z(b_2c_1-b_1c_2)=0;$

$$\therefore \quad x(a_1b_2-a_2b_1)=z(b_1c_2-b_2c_1);$$

$$\therefore \quad \frac{x}{b_1c_2-b_2c_1}=\frac{z}{a_1b_2-a_2b_1}, \qquad (1)$$

$$\text{i.e.} \quad \frac{x}{z}=\frac{b_1c_2-b_2c_1}{a_1b_2-a_2b_1}. \qquad (2)$$

By eliminating x from the original equations, we get

$$\frac{z}{a_1b_2 - a_2b_1} = \frac{-y}{a_1c_2 - a_2c_1}. \quad \text{.......} \quad \text{........} \quad (3)$$

Hence we have the ratio $\frac{y}{z}$.

Results (1) and (3) combined read thus:

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{z}{a_1b_2 - a_2b_1}.$$

This is easily remembered in the following manner:

Write down the coefficients, omitting the x, y and z, thus:

$$\begin{array}{ccc} a_1, & b_1, & c_1, \\ & \searrow & \\ & \nearrow & \\ a_2, & b_2, & c_2. \end{array}$$

To obtain the denominator of x, imagine the a column erased, and take the products of the b's and c's crossways as indicated, the downward arrow being accompanied by a + sign and the upward by a − sign.

To obtain the denominator of $-y$, imagine the b column erased, and proceed as before.

To obtain the denominator of z, imagine the c column erased, and proceed as before.

This method is called **Cross Multiplication.**

Simultaneous equations can be solved by this method: for by putting $z = 1$, we find the equations

$$a_1x + b_1y + c_1 = 0, \quad a_2x + b_2y + c_2 = 0$$

solved in the following form:

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{1}{a_1b_2 - a_2b_1}.$$

Example. Find $\frac{x}{z}$ and $\frac{y}{z}$ from the equations

$$4x - 6y - 24z = 0, \qquad 3x + 7y + 5z = 0.$$

$$\frac{x}{-6\times5+7\times24} = \frac{-y}{4\times5+3\times24} = \frac{z}{4\times7+3\times6}.$$

$$\therefore \frac{x}{3\times46} = \frac{-y}{2\times46} = \frac{z}{46};$$

$$\therefore \frac{x}{z} = 3, \quad \frac{y}{z} = -2.$$

232. To eliminate three unknowns it would in general be necessary to have four equations; but from the three equations

$$a_1x + b_1y + c_1z = 0, \qquad (1)$$

$$a_2x + b_2y + c_2z = 0, \qquad (2)$$

$$a_3x + b_3y + c_3z = 0, \qquad (3)$$

it is possible to eliminate x, y, z; for we are really only eliminating two ratios between them.

From (1) and (2),

$$\frac{x}{b_1c_2 - b_2c_1} = \frac{-y}{a_1c_2 - a_2c_1} = \frac{z}{a_1b_2 - a_2b_1} = k \text{ suppose.}$$

By substituting in (3), $k(b_1c_2 - b_2c_1)$ for x, and similar expression for y and z, and dividing by k, we obtain

$$a_3(b_1c_2 - b_2c_1) + b_3(a_2c_1 - a_1c_2) + c_3(a_1b_2 - a_2b_1) = 0.$$

Examples. XXXVIII. a.

1. By means of squared paper compare the ratio $\frac{11}{19}$ with $\frac{17}{30}, \frac{7}{8}, \frac{14}{23}, \frac{7}{11}, \frac{33}{57}$.

[Join the points (0, 0), (19, 11) and produce this line. Observe whether the point (30, 17), for example, is above or below this line.]

2. Find the ratio of 2 lbs. 6 oz. to 3 lbs. 9 oz.

3. Find the ratio of £2. 5s. 6d. to £4. 0s. 6d.

4. Express as a decimal the ratio 1 inch : 1 cm., if 1 m. = 39·37 inches.

5. What value of x will make $\frac{27+x}{37+x}$ equal $\frac{2}{3}$?

6. If $\frac{5x-4y}{3x-2y} = 4$, find the ratio of x to y.

7. If $\frac{x}{5} = \frac{y}{8}$, find the ratio of $x+5$ to $y+8$.

8. $\frac{x}{y} = \frac{3}{5}$. Find the value of $\frac{x+y}{y-x}$.

9. $3x^2 - 10xy + 3y^2 = 0$. Find $\frac{x}{y}$.

10. $\frac{x+6}{x+15}$ = the duplicate ratio of 4 to 5. Find x.

11. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that $\frac{3a^3 + 4a^2c + 5c^2e}{3b^3 + 4b^2d + 5d^2f} = \frac{a^3}{b^3}$.

12. $\frac{a}{x+y} = \frac{b}{y-z} = \frac{c}{z+x}$. Prove that $a = b + c$.

13. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that $\frac{a^4 + 5c^3e + e^4}{b^4 + 5d^3f + f^4} = \frac{a^2c^2}{b^2d^2}$.

14. Two numbers are in the ratio 3 : 4, and when each is increased by 7 they have the ratio 4 : 5. Find them.

15. Find two numbers in the ratio 5 : 4 such that when each is diminished by 5 they shall be in the ratio 4 : 3.

16. Find two numbers whose sum is 85 and whose ratio is 8 : 9.

17. Divide 92 so that the two parts may be in the ratio 8 : 15.

18. Divide 65 into two parts so that $\frac{3}{5}$ of one may be $\frac{5}{9}$ of the other.

19. The ratio of a rectangle to the square on its diagonal is 6 : 13. Find the ratio of the sides.

20. The ratio of A's age to B's is 5 : 3. 28 years ago it was 4 : 1. How old is A?

21. If $4 : 3 =$ the subduplicate ratio of $x+6$ to $x+2$, find x.

22. If $\frac{a}{b} > \frac{c}{d}$, then $\left(\frac{a^2+c^2}{b^2+d^2}\right)^{\frac{1}{2}} < \frac{a}{b}$ and $> \frac{c}{d}$.

23. Find the least integer which, added to each term of 9 : 17, gives a ratio greater than $\frac{5}{8}$.

24. Find the least integer which, added to each term of 25 : 12, gives a ratio less than $\frac{8}{5}$.

25. What quantity must be added to each term of the ratio $a : b$ to make it equal to $c : d$?

26. If $\frac{x-y}{y+z} = \frac{y-z}{x+y} = \frac{x+z}{x-z}$, each of these ratios $= \frac{x}{x+y}$.

27. Find a ratio intermediate between $\frac{21}{29}$ and $\frac{65}{89}$.

28. If $\frac{a}{b} = \frac{b}{c} = \frac{c}{d}$, prove that each of these ratios $= \frac{ab+bc+cd}{b^2+c^2+d^2}$, and that $\frac{a}{c} = \frac{a^2+b^2+c^2}{b^2+c^2+d^2}$.

29. On squared paper represent the ratios $\frac{3}{5}$, $\frac{5}{9}$, $\frac{6}{11}$, and see which is greatest and which least.

30. By observing where the hypotenuse of each cuts the ordinate whose abscissa is 10, find the value of each of the above ratios in a decimal form.

31. Draw the ratio formed by adding 6 to the numerator and denominator of $\frac{15}{16}$. Compare it with $\frac{15}{16}$.

32. **Show graphically** that, if $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ are all equal, the ratio $\frac{a+c+e}{b+d+f}$ is equal to any one of them, by taking a, c, e for ordinates and b, d, f for abscissae.

33. **Show graphically** that, if $\frac{a}{b}$, $\frac{c}{d}$, $\frac{e}{f}$ are unequal, $\frac{a+c+e}{b+d+f}$ lies in value between the greatest and least of them.

34. Which is the greater, $\frac{x+y}{y}$ or $\frac{4x}{x+y}$?

35. If an object of height h at a distance d from the observer subtends a *small* angle of A degrees at his position, it may be proved that roughly $h = \text{A}d/57{\cdot}3$. Use this to find the height of a tower which subtends an angle of 9° at a point 170 yds. away.

36. If $\frac{(p-1)ab}{pb-a}$ be taken away from each member of the ratio $\frac{a}{b}$, the new ratio is $\frac{a}{pb}$.

37. A and B trade with different sums: A gains £200, B loses £50, and now A's stock : B's = 4 : 1; but if A had gained £100, and B lost £85, their stocks would have been as 60 to 13. Find what each had originally.

38. Construct a scale of feet for a drawing in which 10 ft. 6 in. is represented by $3\frac{1}{2}$ inches.

What ratio does the area of the drawing bear to the area of the figure represented?

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, prove the following (39 to 43):

39. $\frac{3a^2-5ce+7e^2}{3b^2-5df+7f^2}=\frac{ace}{bdf}\times\frac{4b+d}{4a+c}$.

40. $\frac{ae+2c^2}{bf+2d^2}=\frac{a^2+c^2}{b^2+d^2}$.

41. $\frac{la+mc+ne}{lb+md+nf}=\frac{la-mc+ne}{lb-md+nf}=\frac{a}{b}$.

42. $\frac{a^3+c^3}{b^3+d^3}=\frac{ace}{bdf}$.

43. $\sqrt[3]{\frac{pa^2c+qade^2+re^3}{pb^2d+qbdf^2+rf^3}}=\frac{a}{b}$.

44. If $\frac{a}{b}=\frac{3}{5}$, find the value of $\frac{7a+2b}{4a+10b}$.

45. If $\frac{a}{b}=\frac{3}{4}$, and $\frac{c}{d}=\frac{5}{8}$, find the value of $\frac{3ac+5bd}{4ac+8bd}$.

46. If $\frac{2x+6y}{5x+7y}=\frac{4}{7}$, find $\frac{x}{y}$.

47. If $\frac{a+b}{a-b}=\frac{c+d}{c-d}$, prove that $ad=bc$.

48. If $\frac{a}{b}=\frac{c}{d}$, prove that $\frac{a+b}{b}=\frac{c+d}{d}$ and $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

49. If $\frac{a}{b}=\frac{b}{c}$, prove that $a^3+c^3=\left(a+\frac{b^2}{a}\right)(a^2-b^2+c^2)$.

50. If C, F be the readings of any temperature in Centigrade and Fahrenheit scales respectively, prove that $C+40=\frac{5}{9}(F+40)$.

What is the Centigrade reading which corresponds to 41° Fahrenheit?

51. If $\frac{x+2y}{3x-z}=\frac{2x-z}{3y+x}=\frac{5y-3x+z}{4y-4x+z}$, prove that each of these ratios $=1$.

52. If $\frac{x}{a}=\frac{y}{b+c}=\frac{z}{b+c-a}$, prove that $x-y+z=0$.

53. If $\frac{a}{x-y}=\frac{b}{y-z}=\frac{c}{z-x}$ prove that $a+b+c=0$.

54. If $\frac{a}{b}=\frac{b}{c}$, prove that $\frac{a}{c}=\frac{a^2}{b^2}=\frac{a^2+b^2}{b^2+c^2}$.

55. If $\frac{mx+ny}{ma+nb}=\frac{px+qy}{pa+qb}$, each fraction $=\frac{x}{a}=\frac{y}{b}$, if mq and np are unequal.

56. At present A's age is to B's age as 5 to 2, but in 30 years' time the ratio will be 35 : 23. Find their ages.

57. If $4x^2+10y^2=7y(x+y)$, what is the ratio of x to y?

58. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $a+3c+2e:a-e=b+3d+2f:b-f$.

59. Find $x:y$ from the equations $\left.\begin{aligned}3x-4y-7z&=0\\3x+4y-17z&=0\end{aligned}\right\}.$

60. Find $x:z$ from the equations $\left.\begin{aligned}3x-4y-z&=0\\6x+5y-8z&=0\end{aligned}\right\}.$

61. Find $x:y$ and $x:z$ from $\left.\begin{aligned}7x-6y+59z&=0\\3x-8y+47z&=0\end{aligned}\right\}.$

62. Eliminate x, y, z from the equations $\left.\begin{aligned}ax+cy+bz&=0\\cx+by+az&=0\\bx+ay+cz&=0\end{aligned}\right\}.$

63. Eliminate x, y, z from the equations $\left.\begin{aligned}ax+hy+gz&=0\\hx+by+fz&=0\\gx+fy+cz&=0\end{aligned}\right\}.$

64. A sum of money is divided into two parts in the ratio $x:y$. A and B divide between themselves the first part in the ratio $a:b$ and the second part in the ratio $c:d$. If they receive equal amounts, find the ratio of x to y.

65. On a certain map a road 1320 yds. long is represented by $2\frac{3}{4}$ inches. Determine the scale of the map. What area on the map would represent $\frac{1}{10}$ sq. mile?

66. Find the ratio of x to y from the equation $2x^2-9xy+10y^2=0$.

67. Two vessels contain mixtures of wine and water in the ratios of 8 to 3 and 5 to 1 respectively. In what ratio must liquid be drawn from each vessel to give a mixture in the ratio of 4 to 1?

68. In a certain examination the number of those who passed was 3 times the number of those who failed. If there had been 16 fewer candidates and if 6 more had failed, the numbers would have been as 2 to 1. Find the number of candidates.

69. If $\frac{bx-ay}{cy-az}=\frac{cx-az}{by-ax}=\frac{z+y}{x+z}$, each of these ratios $=\frac{x}{y}$, unless $b+c=0$.

70. Two men set out at the same time from A and B along a road ABC, both going in the direction BC. The hinder man travels at $\frac{4}{3}$ of the pace of the other and overtakes him at a point 10 miles from B. Find the distance AB. Solve this question also graphically.

71. The marks gained in an examination-paper for which the maximum was 65 were 53, 42, 37. Find by a diagram what these would be if the maximum were 100.

72. A quantity of milk is increased in the ratio 4 : 5 by watering, and then 3 gallons are sold: the rest by being mixed with 3 quarts of water is increased in the ratio 6 : 7. How many gallons of milk were there at first?

73. Two vessels A and B contain mixtures of water and wine, A in the ratio 2 : 3, B in the ratio 3 : 7. What quantities must be taken from A and B respectively to form a mixture which shall consist of 5 gallons of water and 11 of wine?

PROPORTION.

233. The equality of two ratios forms a **proportion.** Thus a, b, c, d are in proportion if $\frac{a}{b}=\frac{c}{d}$.

Quantities are in **continued proportion,** if the ratio of the 1st to the 2nd = the ratio of the 2nd to the 3rd = the ratio of the 3rd to the 4th, and so on;

$$e.g.\ \tfrac{1}{2}=\tfrac{2}{4}=\tfrac{4}{8}=\ldots.$$

$\therefore$ 1, 2, 4, 8 ... are in *continued* proportion.

If $\frac{a}{b}=\frac{c}{d}$, $ad=bc$; that is, *in any proportion the product of the extremes = the product of the means.*

If $\frac{a}{b}=\frac{b}{c}$, $ac=b^2$; *i.e. if three quantities are in continued proportion, the product of the* 1*st and* 3*rd = the square on the* 2*nd.*

In this case b is said to be a **mean proportional** between a and c, and c a **third proportional** to a and b.

If $\frac{a}{b}=\frac{b}{c}$, then $\frac{a}{c}=\frac{a}{b}\cdot\frac{b}{c}=\frac{a^2}{b^2}$; *i.e. if three magnitudes are in continued proportion the ratio of the* 1*st to the* 3*rd is the duplicate ratio of the* 1*st to the* 2*nd.* (See Art. 225.)

If $\frac{a}{b}=\frac{c}{d}$, the following results are important and easily deducible:

(1) $\frac{\mathbf{a}}{\mathbf{c}}=\frac{\mathbf{b}}{\mathbf{d}}$. **(Alternando.)** (2) $\frac{\mathbf{b}}{\mathbf{a}}=\frac{\mathbf{d}}{\mathbf{c}}$. **(Invertendo.)**

(3) $\frac{\mathbf{a}}{\mathbf{b}}+1=\frac{\mathbf{c}}{\mathbf{d}}+1$, or $\frac{\mathbf{a}+\mathbf{b}}{\mathbf{b}}=\frac{\mathbf{c}+\mathbf{d}}{\mathbf{d}}$. **(Componendo.)**

(4) $\frac{\mathbf{a}}{\mathbf{b}}-1=\frac{\mathbf{c}}{\mathbf{d}}-1$, or $\frac{\mathbf{a}-\mathbf{b}}{\mathbf{b}}=\frac{\mathbf{c}-\mathbf{d}}{\mathbf{d}}$. **(Dividendo.)**

(5) $\frac{\mathbf{a}+\mathbf{b}}{\mathbf{a}-\mathbf{b}}=\frac{\mathbf{c}+\mathbf{d}}{\mathbf{c}-\mathbf{d}}$, from (3) and (4). **(Componendo and Dividendo.)**

A large number of questions in proportion may be solved by the 'k' method explained in Art. 226.

Example 1. If a, b, c, d are in continued proportion,

$$(a-c)(b-d)-(a-d)(b-c)=(b-c)^2.$$

Since $\frac{a}{b}=\frac{b}{c}=\frac{c}{d}$, take each of these ratios equal to k;

$$\therefore\ a=bk,\quad b=ck,\quad c=dk;$$

$$\therefore\ (a-c)(b-d)-(a-d)(b-c)$$

$$=ab-bc-ad+cd-ab+bd+ac-cd$$

$$=ac-ad-bc+bd=bk\cdot\frac{b}{k}-bk\cdot\frac{c}{k}-bc+ck\cdot\frac{c}{k}$$

$$=b^2-2bc+c^2=(b-c)^2.$$

Example 2. If $a+b:b=c+d:d$, then $\frac{a^2+b^2}{a^2-b^2}=\frac{c^2+d^2}{c^2-d^2}$.

Since $$\frac{a+b}{b}=\frac{c+d}{d},\quad \frac{a}{b}+1=\frac{c}{d}+1;$$

$$\therefore\ \frac{a}{b}=\frac{c}{d}=k \text{ suppose};\quad \therefore\ a=bk,\quad c=dk;$$

$$\therefore\ \frac{a^2+b^2}{a^2-b^2}=\frac{b^2k^2+b^2}{b^2k^2-b^2}=\frac{k^2+1}{k^2-1}=\frac{d^2k^2+d^2}{d^2k^2-d^2}=\frac{c^2+d^2}{c^2-d^2}.$$

Examples. XXXVIII. b.

1. If $ab=cd$, express this in the form of a proportion.

2. Find a mean proportional to 7 and 63; and a 3rd proportional to $2x$ and $5x^3$.

3. Find a 4th proportional to a, ab, c; to $3x$, $4y$, $9xz$; and to a^3, ab^2c, bc^2.

If $\frac{a}{b}=\frac{c}{d}$, prove the relations 4–7.

4. $\frac{3a^2+c^2}{3b^2+d^2}=\frac{5a^2-2c^2}{5b^2-2d^2}$.

5. $\frac{a^2+ab+b^2}{a^2-ab+b^2}=\frac{c^2+cd+d^2}{c^2-cd+d^2}$.

6. $\frac{la^2+mc^2}{lb^2+md^2}=\frac{ac}{bd}$.

7. $\sqrt[5]{\frac{2ac^4+3c^5}{2bd^4+3d^5}}=\frac{a}{b}$.

8. If $a:b=b:c$, $a^3+b^3=a(a+b)(a-b+c)$, and
$$a^4+a^2c^2+c^4=(a^2+b^2+c^2)(a^2-b^2+c^2).$$

9. Find two numbers such that their sum, product, and difference of squares are proportional to 7, 12, 7.

10. Three numbers are in continued proportion; the middle one is 15 and the sum of the others 50. Find them.

11. Find a third proportional to $\sqrt{3}+1$ and $\sqrt{3}+2$.

12. Find a mean proportional between $\sqrt{5}+\sqrt{2}$ and $\frac{12}{\sqrt{5}-\sqrt{2}}$.

13. If $(a+b+c+d)(a-b-c+d)=(a+b-c-d)(a-b+c-d)$, then a, b, c, d are in proportion.

If $a:b=b:c$, prove the following (14, 15, 16):

14. $a-b:b-c=b:c$.

15. $a:c=a^2+b^2:b^2+c^2$.

16. $(a+b+c)(b-c)=ab-c^2$.

17. If $a:b=b:c$, and if $b-c=\dfrac{2b}{a}$, prove that $a-c=\dfrac{2(a+b)}{a}$.

18. If $a^2-b^2=(x+y)^2$, put this in the form of a proportion.

19. If a, x, $a-x$ are in continued proportion, find x. Give also a geometrical construction.

20. If $a-b:b-c=b:c$, then a, b, c are in continued proportion.

21. The components of gunpowder are:—nitre 75 per cent., charcoal 15 per cent., and sulphur 10 per cent. How many grams of each (to the nearest gram) are needed to make a pound (454 grams) of gunpowder?

22. Find 2 numbers such that their sum, their difference, and the sum of their squares are proportional to 5, 3, 51.

23. Given $a+b:a-b=c+d:c-d$, express the ratio

$$\sqrt{a^2+pab+b^2}:\sqrt{c^2+pcd+d^2}$$

in terms of a and c alone.

24. If $x+2y:a+3b=y+3x:a+4b$, prove that $x:y=a+5b:2a+5b$, and that $y+2x:x+3y=4a+15b:7a+20b$.

25. If $p:q=a^2:b^2$, and $a:b=(a+p)^{\frac{1}{2}}:(a-q)^{\frac{1}{2}}$, find in terms of a and p the value of $\dfrac{p-q}{q}$.

26. If $b(a-c):c(b-d)=a-b:c-d$, then either $b=c$ or $ad=bc$.

27. If $a:b=c:d$, then $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

28. If a, b, c, d are in continued proportion,

$$a^3+b^3:b^3+c^3=b^3+c^3:c^3+d^3.$$

29. If $\dfrac{a}{b}=\dfrac{c}{d}$, $\left(\dfrac{1}{a}+\dfrac{1}{d}\right)\left(\dfrac{1}{b}+\dfrac{1}{c}\right)=\dfrac{(a-b)(a-c)}{abc}$.

30. If a, b, c are in continued proportion,

$$\frac{a+b+c}{a-b+c}=\frac{(a+b+c)^2}{a^2+b^2+c^2}.$$

31. If $a^2+c^2:ab+cd=ab+cd:b^2+d^2$, prove that $a:b=c:d$.

32. If $a:b=c:d$, $\dfrac{1}{ma}+\dfrac{1}{nb}+\dfrac{1}{pc}+\dfrac{1}{qd}=\dfrac{1}{bc}\left(\dfrac{a}{q}+\dfrac{b}{p}+\dfrac{c}{n}+\dfrac{d}{m}\right)$.

VARIATION.

234. When it is said that x varies as y (written $x \propto y$), it is meant that, however x and y may alter their values, the ratio $x:y$ remains unchanged.

$\therefore$ if $x \propto y$, $\dfrac{x}{y}=$ a constant ratio $=m$ suppose.

Thus if $x \propto y$, $x = my$.

E.g. in a circle the circumference $\propto$ the diameter;

$\therefore$ circumference $\div$ diameter = a constant.

This constant is $3{\cdot}14159\ldots$ and is denoted by π.

$\therefore$ circumference $= 2\pi r$.

235. Variation is a functional way of expressing proportionality.

When we say that y varies as x, we mean that y is proportional to x; *i.e.* y is such a function of x that any change in x produces a proportional change in y.

Thus the symbol $\propto$ means "is proportional to."

If $y \propto x$, and when x has a definite value m, y takes a definite value n, then x, y, m, n are so connected that

$$x : y = m : n.$$

236. A statement of the following sort is commonly made:— "if y denotes the distance travelled by a man walking uniformly, and x the time he has been walking, $y \propto x$, *i.e.* y/x is a constant ratio."

Here the distance and time are not quantities of the same kind, and therefore cannot have a ratio; but y and x are *numbers*, y being the number of units of distance, and x the number of units of time.

y may be the *number* of miles walked, x the *number* of hours.

$\therefore$ y/x is an intelligible ratio.

If it is found that, when 4 hours have elapsed, the distance is 12 miles, $y : x = 12 : 4$; $\therefore$ $y = 3x$.

If we give x any other special value 5, the relation is still $y = 3x$; $\therefore$ $y = 3 \times 5 = 15$.

237. If $y = \frac{k}{x}$, y is proportional to $\frac{1}{x}$, *i.e.* y **varies inversely** as x.

If $y = mx + nx^2$, where m and n are constants, y is a function of x consisting of two terms, one proportional to x, the other proportional to x^2.

If $y=kxz$, where k is a constant and x, z are variables, y is conjointly proportional to x and z, or y is said to vary conjointly as x and z.

If $y=\frac{kx}{z}$, y varies directly as x and inversely as z.

If $y=\frac{k}{x^2}$, y is said to vary inversely as the square of x.

The law of gravitation furnishes an example: for the attraction of the earth on an external object varies inversely as the square of the object's distance from the centre of the earth. If the distance of the object were doubled, the attraction on it would be multiplied by $\frac{1}{4}$; if the distance were trebled, the attraction would be $\frac{1}{9}$ of what it was.

If $y \propto x$, and we express this by $y=mx$, the constant m is called the *constant of the variation*. If at the same time $y \propto z$, we may put $y=nz$, where n is a constant. We must not put $y=mz$, because the constant of the variation is not necessarily the same in both cases.

238. If y is a function of x, the graph of this may be drawn.

When $y \propto x$, $y=mx$.

The graph of this is a straight line through the origin.

When $y \propto x^2$, $y=mx^2$.

The graph of this is a parabola. (See Art. 133.)

When $y \propto \frac{1}{x}$, $xy=m$.

The graph of this is a hyperbola. (See Art. 166.)

239. *If $x \propto y$, and $y \propto z$, then $x \propto z$.*

Since $x \propto y$, $\frac{x}{y}=m$.

Since $y \propto z$, $\frac{y}{z}=n$.

$$\therefore \text{ by multiplication } \frac{x}{z}=mn=\text{a constant};$$

$$x \propto z.$$

240. *If $x \propto y$ when z is constant, and $x \propto z$ when y is constant, then $x \propto yz$ when both y and z are variable.*

[The meaning of this is most easily understood from an example. In a triangle the area $\propto$ the base when the altitude is constant, and varies as the altitude when the base is constant. When both base and altitude are variable, the area $\propto$ base $\times$ altitude.]

Proof. $x \propto y$ when z is constant;

$\therefore$ $x = my$ (where m is independent of y). ....(1)

But $x \propto z$ when y is constant;

$\therefore$ $my \propto z$ when y is constant;

$\therefore$ $m \propto z$;

$\therefore$ $m = nz$ (where n is independent of z). ....(2)

Also n, being a factor of m, is independent of y.

From (2) substitute in (1).

$\therefore$ $x = nzy$ (where n is independent of y and z).

$\therefore$ $x \propto yz$.

Example 1. If $x \propto y$, and $x = 10$ when $y = 3$, find y when $x = \frac{15}{2}$.

Here $x = my$, m being a constant. .... (1)

The statement, that $x = 10$ when $y = 3$, enables us to find m.

For from (1), $10 = m \times 3$; $\therefore$ $m = \frac{10}{3}$;

$\therefore$ the relation between x and y is $x = \frac{10}{3}y$;

$\therefore$ when $x = \dfrac{15}{2}$, we have $\dfrac{15}{2} = \dfrac{10y}{3}$;

$y = \frac{45}{20} = \frac{9}{4}$.

Example 2. If 6 horses can plough $17\frac{1}{2}$ acres in 4 days, how many acres will 54 horses plough in $2\frac{1}{4}$ days?

Denoting the number of horses by H, of acres by A, and of days by D, we know that A $\propto$ H when D is given, and A $\propto$ D when H is given.

$\therefore$ $\mathsf{A} \propto \mathsf{DH}$;

$\therefore$ $\mathsf{A} = m\mathsf{DH}$.

The statement is "6 horses plough $17\frac{1}{2}$ acres in 4 days."

This gives us $\frac{35}{2} = m \times 4 \times 6$;

$\therefore$ $m = \frac{35}{48}$;

$\therefore$ $\mathsf{A} = \frac{35}{48}\mathsf{DH}$;

when $\mathsf{D} = \frac{9}{4}$ and $\mathsf{H} = 54$,

the number of acres $= \frac{35}{48} \times \frac{9}{4} \times 54 = 88\frac{19}{32}$

Example 3. The time of one swing of a simple pendulum $\propto$ the square root of its length. If a pendulum of length 17·44 cm. makes 105 beats in 44 seconds, what is the length of a pendulum which beats exactly once in a second?

If t be the time of a beat in seconds, l the length in cm.,

we have $$t = k\sqrt{l}.$$

When $$l = 17\cdot44, \quad t = \frac{44}{105}.$$

$$\therefore \frac{44}{105} = k\sqrt{17\cdot44}. \quad \therefore k = \frac{44}{105 \times \sqrt{17\cdot44}}.$$

We require the value of l when $t = 1$.

$$\sqrt{l} = \frac{t}{k} = \frac{1}{k}.$$

$$l = \frac{1}{k^2} = \frac{105^2 \times 17\cdot44}{44^2}$$

$$= \frac{105 \times 105 \times 1\cdot09 \times 16}{121 \times 16} = \frac{105 \times 105 \times 1\cdot09}{121}$$

$$= \frac{12017\cdot25}{121} = 99\cdot32 \text{ cm.}$$

241. Instances of Variation are of frequent occurrence in the subject of Physics. Some are quoted here:

(*a*) The pressure of a given mass of gas is directly proportional to the temperature measured from $-273°$ C., and inversely proportional to the volume. $\left[p \propto \frac{T}{V}.\right]$

(*b*) The pressure at any point of a heavy fluid is proportional to the depth of the point. [$p \propto d$, where d is the depth.]

(*c*) The tension of a stretched elastic string is proportional to the extension.

If T be the tension, l the original length, l' the stretched length, then $l' - l =$ the extension.

$$T \propto (l' - l) = k(l' - l).$$

Observe that the tension is proportional to the *increase of length*, not to the stretched length.

Example. A string 4 feet long is stretched to 5 feet by a weight of 3 lbs. To what length will a weight of $5\frac{1}{2}$ lbs. stretch it?

When the tension is 3, the extension $= 5 - 4 = 1$.

But $$T = k(l' - l).$$

$$\therefore 3 = k(5 - 4) = k.$$

$$\therefore T = 3(l' - l).$$

When $T=5\frac{1}{2}$, $\quad l'-l=\frac{T}{3}=\frac{11}{2}\div 3=\frac{11}{6}$.

$$\therefore\ l'=l+\frac{11}{6}=4+\frac{11}{6}=5\tfrac{5}{6}.$$

The stretched length is 5 ft. 10 inches.

(*d*) *Ohm's Law.*—The resistance of a wire of given material to the passage of an electric current is directly proportional to the length of the wire, and inversely proportional to the area of its cross section.

Example. If the resistance of a copper wire 1 kilometre in length and 1 sq. millimetre in section is 16·42 ohms, calculate the resistance of a copper wire 1 sq. cm. in section, and 480 kilometres in length.

R = the resistance $=\dfrac{k\,.\,l}{A}$ (where l = the length, A the area of the section of the wire).

When $l=1$ kilometre, and $A=1$ sq. mm., $R=16·42$;

$$\therefore\ 16·42=k.$$

In the 2nd case $l=480$ kilometres, and $A=100$ sq. mm.;

$$\therefore\ \text{the resistance in the 2nd case}=\frac{16·42\times 480}{100}=1·642\times 48$$
$$=78·8 \text{ ohms approx.}$$

(*e*) The intensity of illumination of a surface varies inversely as the square of the distance from the source.

Examples. XXXVIII. c.

1. If $y\propto x$, and $y=5$ when $x=6$, find the equation between x and y, and draw its graph.

Find y when $x=9$, and find x when $y=3·5$.

Obtain the results graphically and algebraically.

2. If $y\propto\dfrac{1}{x}$, and $y=3$ when $x=2$, find x when $y=21$.

3. If $y\propto\dfrac{1}{x^2}$, and $y=5$ when $x=9$, find y when $x=2$.

4. If $a\propto b$ directly and c inversely, and $a=8$ when $b=10$ and $c=15$, find c when $a=1$ and $b=2$.

5. y consists of a constant term and a term which varies as x. $y=3$ when $x=0$, and $y=7$ when $x=1$. Find the equation between x and y, and determine y graphically when $x=3·5$.

6. $y\propto x$, and $x\propto\dfrac{1}{z^2}$; prove $y^{\frac{1}{2}}z$ = a constant.

7. $a^2+b^2\propto a^2-b^2$; prove that $a+b\propto a-b$.

8. $a+b\propto a-b$; prove that $a^2+b^2\propto ab$.

9. The attraction of the earth on an external object varies inversely as the square of the distance of the object from the earth's centre. Find the apparent weight of a body at a distance of 2000 miles from the earth's surface, supposing it to weigh 1 lb. on the surface of the earth whose radius may be taken to be 4000 miles.

10. If $xy \propto x^2+y^2$, and 3, 4 be contemporaneous values of x and y, express xy in terms of x^2+y^2.

11. If y = the sum of a constant term and a term varying as xy, and $y=-2\frac{1}{3}$ when $x=2$, and $x=-2$ when $y=1$, express y in terms of x.

12. $x \propto \dfrac{z}{y^2}$, and $z^2 \propto \dfrac{y}{x}$; prove $y \propto z \propto \dfrac{1}{x}$.

13. If $x \propto \dfrac{1}{y}+\dfrac{1}{z}$, and $x=3$ when $y=1$ and $z=2$, then $xyz=2(y+z)$.

14. If x varies inversely as $\dfrac{yz}{y-z}$, and is equal to 5 when $y=7$ and $z=2$, then $xyz=14(y-z)$.

15. The wages of 100 men for 6 months amount to £1080. How many men can be employed for 7 months for £453. 12*s.* ?

16. With a capital of £450 a man gains £99 in 11 months. What profit does he make in 10 months on a capital of £1000?

17. A garrison of 1500 men has just provisions enough to allow 26 oz. of bread a day to each man for 38 days. The garrison is increased by 400 men. How many ounces of bread must be assigned to each man to prolong the siege for 27 days longer?

18. A sum of money at simple interest amounts to £688 when the rate is $2\frac{1}{2}$ per cent. and the time 3 years. What would be the amount if the rate were $3\frac{1}{4}$ per cent. and the time $2\frac{1}{3}$ years?

19. The pressure of wind on a plane surface $\propto$ the area of the surface and the square of the wind's velocity. The pressure on a sq. foot is 1 lb. when the wind is moving 15 miles an hour. Find the velocity of the wind when the pressure on a sq. yard is 16 lb.

20. If the wages of 15 boys for 4 weeks come to £30, how many boys will £17. 10*s.* hire for 5 weeks?

21. A book which was 12 feet from a light is moved so as to be 3 feet from it. Compare the intensity of illumination with what it was.

22. A surface is illuminated by a certain light at a distance of 2 feet. Where must it be placed to receive twice the intensity of illumination?

23. If 5 men can do a piece of work in a certain time, how many men will perform another piece of work 7 times as great in one-fifth of the time?

24. If it costs £6 to dig a pit 24 ft. deep and 28 sq. ft. in horizontal section, what is the depth of a pit of horizontal section 14 ft. by 9 ft. which costs £9 to dig out?

25. 50 men do a piece of work, working for 12 days at 7 hours a day: how many hours a day must 15 men work in order to do the same amount in 35 days?

26. The expenses of an institution are partly constant and partly proportional to the number of inmates. When the number of inmates is 80, the expenses are £1700, for 90 inmates the expenses for the same length of time are £1850: what are the expenses for 95 inmates?

27. An elastic string whose unstretched length is 1 foot, is stretched to 14 inches by a weight of 7 lb. What weight will stretch it to 15 inches?

28. If a weight of 5 lb. will stretch an elastic string originally 3 inches long to twice its length, what will be its length when stretched by a weight of 4 lb.?

29. The pressure on a horizontal disc immersed in a liquid $\propto$ the depth of the disc and the square of its radius. If the pressure is 600 lb. when the depth is 5 ft. and radius 3 ft., what is the pressure when the depth is 12 ft. and radius $4\frac{1}{2}$ ft.?

30. One surface is illuminated by a light of 8 candle-power at a distance of 10 ft.; another surface by a light of 25 candle-power at a distance of 30 feet. Find the ratio of the intensity of illumination at the two surfaces.

31. The annual expense of a household of 6 persons is £870. Find the expense of 11 persons, supposing £150 of the expense to be constant and the rest to vary as the number of persons.

32. The area of a circle $\propto$ the square of the radius, and the area is 3·14 square metres when the radius is 1 metre. Find the area when the radius is 5 metres. What radius gives an area of 18·84 sq. metres?

33. The distance fallen by a body from rest $\propto$ the square of the time of fall, and a body falls 64 feet in the first 2 secs. How far does it fall in the next 3 secs.?

34. The value of a diamond $\propto$ the square of its weight, and a diamond of 3 carats is worth £8; find the value of one of the same quality weighing 4 carats.

35. y consists of a constant term and a term varying as x. When $x=2$, $y=26$, and when $x=3$, $y=63$. Find y when $x=2·5$: and find x when $y=40$.

36. The distance of the horizon at sea $\propto$ the square root of the height of the eye above sea-level. Find the distance when the eye is at a height of 6 feet, given that it is 9 miles when the height is 54 feet. Find the height of the eye when the distance is 4 miles.

37. The time of vibration of a pendulum $\propto$ the sq. rt. of its length. The length of one which beats seconds is approximately 30 inches. If it is lengthened by 6 inches, find the time of 1 beat.

38. Weight above the earth's surface varies inversely as the square of distance from the centre, below the surface it varies as the distance from the centre. The earth's radius being reckoned 4000 miles, at what distance below the surface is the weight the same as at 100 miles above it?

39. If a mixture of gold and silver, in which $\frac{3}{4}$ is gold, be worth £49, what will be the value of a mixture of equal weight in which $\frac{1}{2}$ is gold, the value of gold being 16 times that of silver?

40. If the carriages in a railway train be all of the same class and always just full; and if the expense of running a train be proportional to the square of the number of carriages; and if a train of 36 carriages just pay the expense of working it; prove that it will be just as profitable to the railway company to run trains of 16 carriages as trains of 20 carriages.

41. The expenses of a household are partly constant and partly vary as the number of inmates. For 6, 8 and 14 persons the expenses are £16. 10*s*., £18, £22. 10*s*. Draw the graph, find the constant term, and the formula for the expense.

42. The pressure of a quantity of gas in a cylinder with a sliding piston is 30 lb. per sq. in. when the piston is 2 feet from the bottom of the cylinder. If the gas is compressed (without changing its temperature) until the piston is 9 inches from the bottom of the cylinder, what pressure does it then exert?

43. Compare the electrical resistances of two copper wires, their lengths being as 3 : 5, and the diameters of their cross-sections as 1 : 4, respectively.

CHAPTER XXXIX.

LOGARITHMS.

242. DEF. **If one number be chosen as base, the logarithm of any number n to this base is the index of the power to which the base must be raised to be equal to n.**

The logarithm of n to the base a is written $\log_a n$.

Thus if $a^p = n$, $p = \log_a n$.

Logarithms calculated to the base 10 are called *common* logarithms.

$$10 = 10^1 \quad \therefore \text{ by definition } \log_{10} 10 = 1,$$
$$100 = 10^2 \quad \therefore \quad ,, \quad \log_{10} 100 = 2,$$
$$1000 = 10^3 \quad \therefore \quad ,, \quad \log_{10} 1000 = 3,$$
$$\cdot 001 = \tfrac{1}{1000} = 10^{-3} \quad \therefore \quad ,, \quad \log_{10} \cdot 001 = -3,$$
$$125 = 5^3 \quad \therefore \quad ,, \quad \log_5 125 = 3,$$
$$\tfrac{1}{16} = 2^{-4} \quad \therefore \quad ,, \quad \log_2 (\tfrac{1}{16}) = -4.$$

243. $a^0 = 1$; $\therefore$ **$\log_a 1 = 0$,** *i.e.*, **whatever the base, the logarithm of 1 is zero.**

244. *To prove* **$\log_a mn = \log_a m + \log_a n$.**

Let $\log_a m = x$, and $\log_a n = y$.

Then $$m = a^x, \quad n = a^y; \quad \text{(by definition)}$$
$$mn = a^x . a^y = a^{x+y}$$
$$\therefore \log_a mn = x + y = \log_a m + \log_a n,$$
i.e. **the logarithm of a product = the sum of the logarithms of its factors.**

Thus $\log 20 = \log 10 + \log 2 = 1 + \log 2$;

$\log 500 = \log 100 + \log 5 = 2 + \log 5$.

Here $\log 2$ means $\log_{10} 2$, and similarly for the others.

245. *To prove* $\log_a \frac{m}{n} = \log_a m - \log_a n$.

Let $\log_a m = x$, $\log_a n = y$.

Then $$m = a^x, \quad n = a^y;$$

$$\therefore \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y};$$

$$\therefore \log_a \frac{m}{n} = x - y = \log_a m - \log_a n,$$

i.e. **the logarithm of a fraction = log numerator − log denominator,** or **the logarithm of a quotient = log dividend − log divisor.**

Thus $\log \cdot 02 = \log \frac{2}{100} = \log 2 - \log 100 = \log 2 - 2$

When common logarithms are used, the base is generally not written.

Thus $\log \cdot 02$ is understood to mean $\log_{10} \cdot 02$.

246. *To prove* $\log_a n^r = r \log_a n$

Let $\log_a n = x$, then $n = a^x$;

$$\therefore n^r = (a^x)^r = a^{rx};$$

$$\therefore \log_a n^r = rx = r \log_a n,$$

i.e. **the logarithm of any power of a number is the product of the logarithm of the number and the index of the power.**

Thus $\log 10000 = 4 \log 10 = 4$; $\log 16 = \log 2^4 = 4 \log 2$,

and $$\log_2 128 = \log_2 2^7 = 7 \log_2 2 = 7.$$

247. *To prove that* $\log_a n = \frac{\log_b n}{\log_b a}$.

Let $n = a^x$, so that $\log_a n = x$.

Also, taking logarithms of both sides to the base b,

$$\log_b n = \log_b a^x = x \log_b a;$$

$$x = \frac{\log_b n}{\log_b a},$$

$$\textit{i.e.} \quad \log_a n = \frac{\log_b n}{\log_b a}.$$ Q.E.D.

Notice that we have changed the base.

If by a stretch of imagination we regard the b's in the above formula as cancelling, as in the fraction $\dfrac{\frac{v}{a}}{\frac{a}{b}}$ (though no such process goes on), it is easily remembered.

Beginners often make a mistake in saying that

$$\frac{\log_b n}{\log_b a} = \log_b n - \log_b a.$$

248. A particular case of the formula of Art. 247 is

$$\log_a b = \frac{\log_b b}{\log_b a} = \frac{1}{\log_b a}.$$

$$\therefore \ \log_a b \,.\, \log_b a = 1.$$

Or we might prove it thus:

Let $\log_a b = x$, so that $a^x = b$.

Taking logs to base b, $x \log_b a = \log_b b = 1$,

$$i.e.\ \log_a b \,.\, \log_b a = 1.$$

Example 1. Simplify $\log_b a \,.\, \log_c b \,.\, \log_d c$.

$\log_b a \,.\, \log_c b = \log_c a$ (by Art. 247, the b's appearing to cancel).

$$\therefore \ \log_b a \,.\, \log_c b \,.\, \log_d c = \log_c a \,.\, \log_d c = \log_d a.$$

Or thus, without assuming Article 247.

Let $\log_b a = x$. $\therefore \ a = b^x$.

Let $\log_c b = y$. $\therefore \ b = c^y$.

Let $\log_d c = z$. $\therefore \ c = d^z$.

Required xyz.

$$a = b^x = (c^y)^x = c^{xy} = (d^z)^{xy} = d^{xyz}.$$

$$\therefore \ \log_d a = xyz = \log_b a \,.\, \log_c b \,.\, \log_d c.$$

Example 2. Simplify $\log_{10} 64 \,.\, \log_5 10 \div \log_5 8$.

The expression $= \log_{10} 64 \cdot \dfrac{\log_5 10}{\log_5 8} = \log_{10} 64 \,.\, \log_8 10$

$$= \log_8 64 = \log_8 8^2 = 2.$$

Characteristic and Mantissa.

249. DEF. *The integral part of a logarithm is called its* **characteristic,** *the decimal part its* **mantissa.**

Example. $\log 20 = 1{\cdot}3010$.

$\therefore$ the *characteristic* of $\log 20$ is unity,

and its *mantissa* is $\cdot 3010$.

If a number, n, lies between 10^p and 10^{p+1}, $n = 10^{p + \text{a decimal}}$.

$\therefore$ its common logarithm is $p +$ a decimal.

$\therefore$ the characteristic of $\log_{10} n$ is p.

Hence, to determine the characteristic of $\log_{10} n$, we only have to find what two *consecutive* powers of 10 the number n lies between.

Example. $125 > 10^2$ and $< 10^3$.

$\log 125 = 2 +$ a decimal

$\therefore$ the characteristic of $\log 125$ is 2.

$2354 > 10^3$ and $< 10^4$.

$\therefore$ the characteristic of $\log 2354$ is 3.

$67 \cdot 04 > 10^1$ and $< 10^2$.

$\therefore$ the characteristic of $\log 67 \cdot 04$ is unity.

Hence we see that:

The characteristic of log n, when n > 1, is one less than the number of integral digits in the number n.

Again, $\log_{10} 1 = 0$. $\therefore$ the logarithm of a number less than 1 is less than 0, *i.e.* it is negative.

We will now show how to find the characteristic of $\log n$ when $n < 1$.

For convenience sake a negative mantissa is made positive by changing the characteristic.

Thus $\qquad \log \cdot 3 = -0 \cdot 5229 = -1 + 0 \cdot 4771,$

which we write thus $\overline{1} \cdot 4771$, the *bar* over the 1 indicating that the mantissa is *positive*, though the rest is negative.

($\overline{2} \cdot 4771$ is read thus "2 bar decimal 4771.")

Example. $\log \cdot 025 = \log \frac{1}{40} = -\log 4 - \log 10 = -1 \cdot 6021 = -2 + (1 - \cdot 6021)$
$= \overline{2} \cdot 3979.$

We will now examine several different cases.

$\cdot 3 = \frac{3}{10}$, which $> \frac{1}{10}$ and < 1,

i.e. it $> 10^{-1}$ and $< 10^0$;

$\therefore \log \cdot 3 = -1 +$ a decimal;

$\therefore$ the characteristic of $\log \cdot 3$ is $\overline{1}$.

$$\cdot 035 > \cdot 01 \text{ and } < \cdot 1,$$

$$\textit{i.e.} \text{ it } > 10^{-2} \text{ and } < 10^{-1};$$

$$\therefore \log \cdot 035 = -2 + \text{a decimal};$$

$$\therefore \text{ the characteristic of } \log \cdot 035 \text{ is } -2.$$

$$\cdot 00003781 > \cdot 00001 \text{ and } < \cdot 0001,$$

$$\textit{i.e.} \text{ it } > 10^{-5} \text{ and } < 10^{-4};$$

$$\therefore \text{ the characteristic of } \log \cdot 00003781 \text{ is } -5.$$

These examples establish the following rule:

If n is any decimal less than unity, the characteristic of log n is negative, and numerically one more than the number of zeros before the first significant figure.

[**Reminder.** 'The significant figures of a number are those which remain when all zeros at the beginning and end have been removed.

Example. The significant figures of ·0032016, and of 32016000 are 32016.]

250. The mantissae of the logarithms of numbers are the same if the numbers have the same significant figures.

This is the same as saying that the logarithms of numbers *whose quotient is any power of* 10 have the same mantissa.

The following examples establish the truth of this statement:

$$\frac{32016000}{\cdot 0032016} = \frac{10^3 \times 32016}{10^{-7} \times 32016} = 10^{10};$$

$$\therefore \log 32016000 - \log \cdot 0032016 = 10,$$

i.e. $\log 32016000$ and $\log \cdot 0032016$ have the same mantissa.

$$\frac{10 \cdot 357}{\cdot 010357} = \frac{1000 \times \cdot 010357}{\cdot 010357} = 10^3;$$

$$\therefore \log 10 \cdot 357 - \log \cdot 010357 = 3,$$

i.e. $\log 10\cdot 357$ and $\log \cdot 010357$ have the same mantissa.

Example. $\log 624 = 2 \cdot 7952;$

$\therefore \log 624000 = 5 \cdot 7952, \quad \log \cdot 0624 = \bar{2} \cdot 7952,$

$\log 6 \cdot 24 = \cdot 7952, \text{ and } \log \cdot 624 = \bar{1} \cdot 7952.$

Examples. XXXIX. a. (*Oral.*)

(*The logarithms are to the base* 10 *unless otherwise indicated.*)

Read off or write down the following logarithms to the base 10:

1. $\log 1000$. **2.** $\log 100{,}000$. **3.** $\log \cdot 01$. **4.** $\log \cdot 0001$.

Read off or write down the following logarithms to the base 2:

5. $\log 64$. **6.** $\log 256$. **7.** $\log 1$. **8.** $\log \frac{1}{2}$. **9.** $\log \frac{1}{16}$.

Read off or write down:

10. $\log_9 81$. **11.** $\log_4 16$. **12.** $\log_3 27$. **13.** $\log_5 125$. **14.** $\log_4 64$.

15. $\log_7 343$. **16.** $\log_2 32$. **17.** $\log_2 8$. **18.** $\log_{10} \cdot 1$

19. $\log_3 81$. **20.** $\log_9 3$. **21.** $\log_{27} 3$. **22.** $\log_3 243$.

23. $\log_5 3125$. **24.** $\log_{25} 5$. **25.** $\log_{125} 5$. **26.** $\log_8 16$.

27. $\log_4 32$. **28.** $\log_8 2$. **29.** $\log_{\frac{1}{2}} 8$. **30.** $\log_{\cdot 10} 10$.

Write down or read off other forms of:

31. $\log abc$. **32.** $\log a^2b^3c^5$. **33.** $\log \frac{ab^2}{cd^2}$.

Write down or read off in terms of $\log 2$ and $\log 3$:

34. $\log 4$. **35.** $\log 6$. **36.** $\log 8$. **37.** $\log 15$. **38.** $\log 18$.

Simplify:

39. $\log 2 + \log 5$. **40.** $\log 2 + \log 3$. **41.** $\log 12 - \log 3$.

42. $\log 12 - 2\log 2$. **43.** $\log 54 - 2\log 3$. **44.** $\log 3000$.

Solve orally the equations:

45. $2^x = 64$. **46.** $2^x \times 3^x = 216$. **47.** $5^x = 3125$.

48. $2^x = \cdot 5$. **49.** $2^x = \cdot 125$.

Write down or read off the characteristic of each of the following:

50. $\log 317$. **51.** $\log 1234$. **52.** $\log 12 \cdot 3$. **53.** $\log \cdot 623$.

54. $\log \cdot 048$. **55.** $\log 6043$. **56.** $\log 801$. **57.** $\log \cdot 0017$.

58. $\log \cdot 32$. **59.** $\log 3 \cdot 241$. **60.** $\log \cdot 000926$.

61. Given $a = 10^5 . b$, find $\log a \sim \log b$.

62. Given $\log_{10} 5 = x$, find $\log_{10} \cdot 005$.

Simplify:

63. $\log_3 8 \div \log_3 2$. **64.** $\log_3 4 \times \log_4 3$. **65.** $\log_3 4 \times \log_2 3$.

66. Explain why $\log 3251$ and $\log 9999$ have the same characteristic.

67. Prove that $\log 723$ and $\log 7 \cdot 23$ have the same mantissa.

68. In what other form can $\log (a^2 - b^2)$ be put?

69. What are the significant figures in 203500?

70. .. .. $20 \cdot 35$?

71. .... :. .. .. $\cdot 00516$?

72. ... . $\cdot 01308$?

73. Given $\log 2592 = 3 \cdot 4136$, write down $\log 2 \cdot 592$.

74. ... ... $\log 259200$.

75. .... $\log \cdot 002592$.

76. Express $\log 5^3 . 3^4 . 7^2$ in terms of $\log 2$, $\log 3$, and $\log 7$.

77. Read off $\log 3762 - \log 37 \cdot 62$ without tables.

78. .... $\log 134 \cdot 5 - \log 0 \cdot 1345$ ....

79. .... $\log 7 \cdot 8 - \log 0 \cdot 78$ ....

80. .... $\log 543 - \log \cdot 00543$ ....

Given that $\log 2 = \cdot 3010$, read off the value of

81. $\log 20$. 82. $\log 2000$. 83. $\log \cdot 2$.

84. $\log \cdot 0002$. 85. $\log 2 \times 10^5$. 86. $\log \left(\frac{2}{10^3} \right)$.

Given that $\log 2364 = 3 \cdot 3736$, read off the value of

87. $\log 2 \cdot 364$. 88. $\log 236 \cdot 4$. 89. $\log 236400$.

90. $\log \cdot 2364$. 91. $\log \cdot 002364$. 92. $\log (2 \cdot 364 \times 10^5)$.

251. Care must be taken in dealing with numbers in the form $\bar{2} \cdot 0345$.

Example 1. $\bar{4} \cdot 4771 \times 5 = (-4 + \cdot 4771) \times 5 = -20 + 2 \cdot 3855$
$= \overline{18} \cdot 3855$.

Example 2. Divide $\bar{4} \cdot 4771$ by 5.

The characteristic -4 is not exactly divisible by 5.

Therefore we make it so by writing $\bar{4} \cdot 4771$ in the form $-5 + 1 \cdot 4771$.

Hence $\frac{\bar{4} \cdot 4771}{5} = \frac{-5 + 1 \cdot 4771}{5} = -1 + \cdot 2954 = \bar{1} \cdot 2954$.

Example 3. Add together $\bar{3} \cdot 4771$, $6 \cdot 4812$, $\bar{9} \cdot 9023$.

$$\begin{array}{r} \bar{3} \cdot 4771 \\ 6 \cdot 4812 \\ \underline{\bar{9} \cdot 9023} \\ \underline{\underline{\bar{5} \cdot 8606}} \end{array}$$

To the left of the decimal point we have $-12 + 6 + 1$ (carried) $= -5$.

Example 4. Subtract $\bar{4} \cdot 6917$ from $\cdot 0312$.

$$\begin{array}{r} \cdot 0312 \\ \underline{\bar{4} \cdot 6917} \\ \underline{\underline{3 \cdot 3395}} \end{array}$$ (Check by adding the 2nd and 3rd lines.)

Or thus: $\cdot 0312 - (\bar{4} \cdot 6917) = \cdot 0312 + 4 - \cdot 6917$
$= 4 \cdot 0312 - \cdot 6917 = 3 \cdot 3395$.

Example 5. Simplify $2\log\frac{16}{15}+3\log\frac{5}{2}+\log\frac{9}{80}$.

$$\text{The expression}=\log\left(\frac{16^2}{15^2}\cdot\frac{5^3}{2^3}\cdot\frac{9}{80}\right)$$
$$=\log\left(\frac{2^8}{3^2.5^2}\cdot\frac{5^3}{2^3}\cdot\frac{3^2}{5.2^4}\right)$$
$$=\log\left(\frac{2^8.3^2.5^3}{2^7.3^2.5^3}\right)=\log 2.$$

Example 6. Given that $\log 5=\cdot6990$, find how many digits there are in 5^9.

$$\log 5^9=9\log 5=9\times\cdot6990=6\cdot2910.$$

The characteristic 6 tells us that there are 7 integral digits in 5^9; and we know that it is entirely integral;

$\therefore$ the number of digits is 7.

Example 7. What power of 3 is nearest to 10^8? ($\log 3=\cdot4771$.)

Let $3^x=10^8$.

Take logarithms of both sides.

Then $x\log 3=8$;

$\therefore\ x=8\div\log 3=8\div\cdot4771=16\cdot8$ nearly.

This is nearer to 17 than to 16;

$\therefore$ the 17^{th} is the nearest power.

Examples. XXXIX. b

1. Add together $1\cdot2864$, $2\cdot1572$, $3\cdot7632$.
2. Add together $5\cdot6391$, $1\cdot5534$, $\cdot7431$.
3. Subtract $1\cdot2345$ from $\bar{2}\cdot6387$.
4. Subtract $\bar{2}\cdot5461$ from $1\cdot0386$.
5. Multiply $\bar{2}\cdot4771$ by 5.
6. Multiply $\bar{3}\cdot6990$ by 2.
7. Divide $\bar{1}\cdot5423$ by 3.
8. Divide $\bar{2}\cdot3184$ by 4.
9. Divide $3\cdot3075$ by 5.
10. How many zeros follow the decimal point in $\cdot2^{20}$?
11. How many digits are there in 2^{20}?
12. .... 2^{25}?
13. Prove that $\log(9^2-3^2)-(\log 9^2-\log 3^2)=\log 8$.
14. Simplify $\log\frac{15}{16}+\log\frac{64}{41}-\log\frac{4}{27}$.
15. Simplify $\frac{1}{2}\log 196+\log 25-2\log 2+\log\frac{3}{7}-\log 3\cdot75$.
16. How many digits are there in 7^{40}?
17. Given $\log 2=\cdot3010$, find what power of 2 is nearest to 10^4.
18. .... 10^7.
19. .... 10^6.
20. Given that $\log_{10}7=\cdot8451$ and $\log_{10}4=\cdot6021$, find $\log_4 7$ to 2 decimal places.

21. Find the value of $\log_b a \,.\, \log_c b \,.\, \log_a c$.

22. .... $\log_{10} 9 \,.\, \log_4 10 \div \log_4 3$.

Solve to 2 decimal places:

23. $6^x = 126$, given $\log 6 = \cdot 7782$ and $\log 126 = 2 \cdot 1004$.

24. $2^x . 3^{5x} = 144$, given $\log 2 = \cdot 3010$ and $\log 3 = \cdot 4771$.

25. $3^{x+1} = 5^{x-1}$, given $\log 3 = \cdot 4771$ and $\log 5 = \cdot 6990$.

The Principle of Proportional Parts.

252. *For numbers differing by small quantities, i.e. by small fractions of themselves, the differences of the logarithms are approximately proportional to the differences of the numbers.*

This principle is most important in the construction, and to some extent in the use, of tables.

If we know $\log 213$ and $\log 214$, by this principle we can find $\log 213 \cdot 7$.

$$\log 213 = 2 \cdot 3284$$

$$\text{and} \quad \log 214 = 2 \cdot 3304.$$

Here the numbers differ by 1, and their logarithms differ by $\cdot 0020$.

If the numbers differed by $\frac{1}{4}$ of 1, their logarithms would differ by $\frac{1}{4}$ of $\cdot 0020$; and similarly for other cases.

If we wish to obtain $\log(213 + x)$ from $\log 213$, we may call it $2 \cdot 3284 + y$, where we recognise that y is the same fraction of $\cdot 0020$ as x is of 1.

Thus $\qquad \log 213 \cdot 7 = \log(213 + \cdot 7) = 2 \cdot 3284 + y,$

where $\qquad \dfrac{y}{\cdot 0020} = \dfrac{\cdot 7}{1}, \quad$ *i.e.* $y = \cdot 0014$.

Hence $\qquad \log 213 \cdot 7 = 2 \cdot 3284 + \cdot 0014 = 2 \cdot 3298.$

253. MATHEMATICAL TABLES.

Logarithms.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	3	6	8	11	14	17	20	22	25
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	3	5	8	11	13	16	18	21	24
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	2	5	7	10	12	15	17	20	22
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	2	5	7	9	12	14	16	19	21
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	2	4	7	9	11	13	16	18	20
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	2	4	6	8	11	13	15	17	19
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	2	4	6	8	10	12	14	16	18
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	2	4	6	8	10	12	14	15	17

The quotation given above from a table of 4-figure logarithms will show the method of reading the logarithm of any number. For instance suppose that log 1756 is required. Look along the line beginning 17 until you reach the figures below the 5 which occurs in the top line. The figures are 2430. For the final 6 take the figures below 6 in the columns on the right of the page, viz. 15. The total result is 2445. The decimal point in the logarithms is not printed, so that in reality the figures are ·2430 and ·0015, giving a total of ·2445 as the mantissa of the logarithm of a number whose significant figures are 1756.

Add the proper characteristic, and the logarithm of 1756 is 3·2445.

Also $\log 17{\cdot}56 = 1{\cdot}2445$ and $\log {\cdot}01756 = \bar{2}{\cdot}2445$;

for the logarithms of numbers have the same mantissa if the numbers themselves have the same significant digits.

By the principle of proportional parts, if we required log 1756·7 we should have to add $\frac{1}{10}$ of the difference for 7; *i.e.* $\frac{{\cdot}0017}{10}$, *i.e.* ·0002.

Thus $\log 1756{\cdot}7 = 3{\cdot}2447$.

Antilogarithms.

254. Def. If x is the logarithm of n, then n is called the **antilogarithm** of x.

The reverse process, that of finding the antilogarithm of a set of figures (*i.e.* the number whose logarithm is the given set of figures), can be accomplished by searching in the columns for the given set of figures or the set next less than these, and making, in the latter case, the proper allowance for the difference by means of the right-hand columns. Labour is saved, however, by using tables of **antilogarithms**, which are read in a similar manner to the tables of logarithms.

It must be remembered that the mantissa only is given in the table and only the **significant digits** of the antilogarithm. The position of the decimal point in the antilogarithm must be determined by the given characteristic.

Antilogarithms.

	0	1	2	3	4	5	6	7	8	9	1	2	3	4	5	6	7	8	9
·19	1549	1552	1556	1560	1563	1567	1570	1574	1578	1581	0	1	1	1	2	2	3	3	3
·20	1585	1589	1592	1596	1600	1603	1607	1611	1614	1618	0	1	1	1	2	2	3	3	3
·21	1622	1626	1629	1633	1637	1641	1644	1648	1652	1656	0	1	1	2	2	2	3	3	3
·22	1660	1663	1667	1671	1675	1679	1683	1687	1690	1694	0	1	1	2	2	2	3	3	3
·23	1698	1702	1706	1710	1714	1718	1722	1726	1730	1734	0	1	1	2	2	3	3	3	4
·24	1738	1742	1746	1750	1754	1758	1762	1766	1770	1774	0	1	1	2	2	2	3	3	4
·25	1778	1782	1786	1791	1795	1799	1803	1807	1811	1816	0	1	1	2	2	2	3	3	4
·26	1820	1824	1828	1832	1837	1841	1845	1849	1854	1858	0	1	1	2	2	3	3	3	4
·27	1862	1866	1871	1875	1879	1884	1888	1892	1897	1901	0	1	1	2	2	3	3	3	4

Find the antilogarithm of ·2445 and of 3·2445.

The significant digits are found by going along the line which begins with ·24 until the column headed by 4 is reached, and adding to the 1754, thus obtained, the figure under 5 in the right-hand columns.

Thus the antilogarithm of ·2445 is 1·756, since the significant digits are 1756 and the characteristic 0 shows that there is *one* integral figure.

So also 3·2445 is the logarithm of 1756.

255. The tables of logarithms may be used in such a manner as to obtain the proper characteristic without direct reference to the rules given for writing down the characteristic. The method itself explains how the characteristic occurs. Any logarithm as it appears in the tables, viz. with 0 for characteristic, is the logarithm of a number containing one integral digit. This may be regarded as the **standard form**; and all numbers, whose logarithms are required, may be expressed in terms of this standard form by multiplying or dividing by some power of 10.

For instance $\log 7{\cdot}253 = 0{\cdot}8605$.

$$\log 7253 = \log(7{\cdot}253 \times 10^3) = {\cdot}8605 + 3 = 3{\cdot}8605.$$

$$\log {\cdot}0007253 = \log(7{\cdot}253 \times 10^{-4}) = {\cdot}8605 - 4 = \bar{4}{\cdot}8605.$$

Before attempting examples involving logarithms, the student should have some oral practice in the use of logarithm and antilogarithm tables.

E.g. **Read off log 62·37, log 620·9, log ·0271, and so on.**

Read off the numbers whose logarithms are

3·235, 1·067, ·0824, $\bar{1}$·6258, and so on.

Example 1. A cube contains 3 c. ft. 904 c. in. Find the length of its edge.

3 c. ft. 904 c. in. = 6088 c. in.

$$\log \sqrt[3]{6088} = \frac{1}{3}\log 6088 = \frac{3\cdot7845}{3} = 1\cdot2615.$$

Looking at the table of antilogarithms, we find that corresponding to ·261 are the figures 1824, and from the columns on the right we see that for the final 5 we must add 2.

$\therefore$ the significant figures required are 1826.

The characteristic 1 (in 1·2612) shows that there are 2 integral figures;

$$\therefore \text{ antilog } 1\cdot2615 = 18\cdot26.$$

$$\therefore \sqrt[3]{6088} = 18\cdot26.$$

$\therefore$ the length of edge is 1 ft. 6·26 in.

After a little practice it would only be necessary to write down

$$\log \sqrt[3]{6088} = \frac{1}{3}\log 6088 = \frac{3\cdot7845}{3} = 1\cdot2615 = \log 18\cdot26.$$

Length of edge = 18·26 in. = 1 ft. 6·26 in.

Example 2. Find by logarithms the product of 2·413 and ·6052.

The logarithm of the product $= \log 2\cdot413 + \log \cdot6052$

$= \cdot3825 + \bar{1}\cdot7819 = \bar{1} + 1\cdot1644$

$= \cdot1644.$

From the table of antilogarithms we find antilog ·1644 = 1·460.

$\therefore$ the required product = 1·460 to 3 decimal places.

This might be worked concisely as follows:

$$2\cdot413 \times \cdot6052 = \text{antilog}\left[\begin{array}{l} \quad\cdot3825 \\ +\bar{1}\cdot7819 \end{array}\right.$$

$$= \text{antilog } \cdot1644 = 1\cdot460.$$

Example 3. Find the value of 2·644 ÷ ·2863.

The logarithm of the quotient $= \log 2\cdot644 - \log \cdot2863$

$= \cdot4223 - \bar{1}\cdot4569 = 1\cdot4223 - \cdot4569$

$= \cdot9654 = \log 9\cdot235.$

$\therefore$ the required quotient = 9·235 (correct to 3 decimal places).

Or, more shortly, $2\cdot644 \div \cdot2863 = \text{antilog}\left[\begin{array}{l} \quad\cdot4223 \\ -\bar{1}\cdot4569 \end{array}\right.$

$$= \text{antilog } \cdot9654 = 9\cdot235.$$

Example 4. Find the value of (£1. 3*s*. 6*d*.) × ·784.

£1. 3*s*. 6*d*. = £1·175.

$$\log(1\cdot175 \times \cdot784) = \log 1\cdot175 + \log \cdot784 = \cdot0701 + \bar{1}\cdot8943$$

$$= \bar{1}\cdot9644 = \log \cdot9212.$$

$\therefore$ the required value = £·9212 = 18*s*. 5*d*.

Example 5. Find the square root of 73.

$$\sqrt{73} = \text{antilog}(\log\sqrt{73}) = \text{antilog}(\tfrac{1}{2}\log 73)$$
$$= \text{antilog}(\tfrac{1}{2}\times 1\cdot 8633) \quad \text{(from tables)}$$
$$= \text{antilog}(\cdot 93165)$$
$$= 8\cdot 544 \quad \text{(from tables)}.$$

Example 6. Given $\log 2810 = 3\cdot 4487$, and $\log 2820 = 3\cdot 4502$, find $\log 28\cdot 16$.

We will first find $\log 2816$. Let $\log 2816 = \log 2810 + x = 3\cdot 4487 + x$.
$\log 2820 - \log 2810 = 3\cdot 4502 - 3\cdot 4487 = \cdot 0015$;
i.e. $\cdot 0015$ is the diff. between the logs when 10 is the diff. between the numbers,
and x .... 6 ....

$$\therefore \frac{x}{\cdot 0015} = \frac{6}{10}, \quad \therefore x = \cdot 0009.$$
$$\therefore \log 2816 = 3\cdot 4487 + \cdot 0009 = 3\cdot 4496.$$
$$\therefore \log 28\cdot 16 = 1\cdot 4496.$$

Example 7. Given $\log 3450 = 3\cdot 5378$, and $\log 3460 = 3\cdot 5391$, find antilog $3\cdot 5388$.

The number required evidently lies between 3450 and 3460.
Let it be $3450 + x$.

$$3\cdot 5391 - 3\cdot 5378 = \cdot 0013,$$
$$3\cdot 5388 - 3\cdot 5378 = \cdot 0010.$$

$\therefore$ by the Principle of Proportional Parts, $\dfrac{x}{10} = \dfrac{\cdot 0010}{\cdot 0013} = \dfrac{10}{13}$

and $x = \dfrac{100}{13} = 8$ to the nearest integer.

$\therefore$ $3458 = \text{antilog}\, 3\cdot 5388$.

Examples. XXXIX. c.

From tables write down

1. $\log 26$. 2. $\log 2600$. 3. $\log 265$. 4. $\log 2658$.
5. $\log 2\cdot 658$. 6. $\log 265\cdot 8$. 7. $\log \cdot 002658$.
8. Given antilog $\cdot 6153 = 4\cdot 124$, find antilog $2\cdot 6153$.

If antilogarithm tables are supplied, they may be used for Examples 9–22. Otherwise, the method of Proportional Parts must be used. See Art. 252, and Example 6 above.

Find the antilogarithms of the following:

9. $\cdot 3851$. 10. $1\cdot 3851$. 11. $2\cdot 3851$. 12. $\bar{2}\cdot 7861$. 13. $3\cdot 8423$.
14. $1\cdot 7621$. 15. $\cdot 7449$. 16. $2\cdot 9022$. 17. $\cdot 9032$. 18. $\bar{2}\cdot 8021$.
19. $\bar{1}\cdot 8591$. 20. $\bar{3}\cdot 4623$. 21. $5\cdot 5400$. 22. $5\cdot 9228$.

Reduce the following (23 to 34) to *standard form*, and find their logarithms.

23. 36840. **24.** 368·4. **25.** 1567. **26.** ·01567.

27. 428·6. **28.** 4286000. **29.** 2113·5. **30.** ·0021135.

31. 3865. **32.** 0·05713. **33.** 7641000. **34.** 0·7648.

35. Find approximately by logarithms the product of 1·414 and 1·732.

36. .... $\sqrt{7}$.

37. . . .. $\sqrt[3]{7}$.

38. Solve the equation $2^x=3$. **39.** Solve the equation $3^{x+4}=405$.

40. Solve $3^x . 2^y=100$, $2^x . 3^y=50$.

41. Plot the curve $y=2^x$ from $x=0$ to $x=6$, using 1 inch for the unit of abscissae, one-tenth for unit of ordinates. Hence find $2^{1\cdot5}$, $2^{3\cdot4}$, $2^{4\cdot5}$.

Find also $\log_2 14$ and $\log_2 23$.

42. Simplify $\log 98^{\frac{1}{2}}+\log 245-\log 17\cdot15$.

43. Simplify $\log 15+\log 105 \quad \log 225$.

44. Find log 81 to the base 243.

45. Prove $\log 20+7\log\frac{15}{16}+5\log\frac{24}{25}+3\log\frac{80}{81}=1$.

Find approximately the results of the following by four figure logarithms:

46. $2\cdot3\times1270$. **47.** $\cdot2413\times6\cdot052$. **48.** $4\cdot951\times2\cdot836$.

49. $\cdot3463\times\cdot3973$. **50.** $\cdot746\times\cdot6235$. **51.** $3\cdot72^4$.

52. $\cdot407\times40\cdot3\times\cdot006$. **53.** $\cdot0438\times937$. **54.** $48\cdot25\div634\cdot9$.

55. $\cdot07644\div147$. **56.** $\cdot86751\div24\cdot3$. **57.** $8\cdot30676\div3596$.

58. $\sqrt{184\cdot12}$. **59.** $\left(\cdot9375\times\cdot018^{\frac{1}{7}}\right)^{\frac{1}{11}}$. **60.** $\sqrt{\dfrac{15\cdot92^{\frac{1}{7}}\times\cdot0182^{\frac{1}{3}}}{\cdot00526\times196^5}}$.

61. $\sqrt[8]{10}$ correct to 3 places. **62.** $3\cdot8^3$. **63.** $\sqrt[3]{68921}$.

64. $\dfrac{1}{\sqrt{\cdot0714}}$. **65.** $\dfrac{1}{\sqrt{\cdot0196}}$. **66.** Solve $4\times2^{2-x}=5^{6x-4}$.

67. Given $\pi=3\cdot142$, find $\frac{1}{\pi}$. **68.** Solve $182^x\times3=5$.

69. Solve $(\frac{4}{7})^x=0\cdot0016$.

70. Calculate the value of $3\cdot74/4\pi$ to 3 decimal places.

71. Calculate the value of $30\cdot26\times18\times8\cdot5\div(13\cdot6\times22)$.

72. Find a fourth proportional to 0·001233, 13·6 and 51, correct to 2 significant figures.

73. The specific gravity of pure alcohol at a certain temperature is stated to be $\dfrac{49\cdot3}{50\cdot7}\div\dfrac{57\cdot09}{42\cdot91}$. Work this out correct to 3 decimal places.

74. In calculating the electric capacity of a certain wire, the following formula occurs: $4{\cdot}2 \times 6087 \times 30{\cdot}48 \div \{4{\cdot}6052 \times (28{\cdot}8)^2 \times 10^3\}$. Work it out to 3 decimal places.

75. By successively taking square roots it may be found that $10^{\frac{1}{2}} = 3{\cdot}1623$, $10^{\frac{1}{4}} = 1{\cdot}778$, $10^{\frac{1}{8}} = 1{\cdot}334$, $10^{\frac{1}{16}} = 1{\cdot}155$, $10^{\frac{1}{32}} = 1{\cdot}074$.

By multiplication obtain the values of $10^{\frac{3}{32}}$ and $10^{\frac{5}{32}}$.

Plot the curve $y = 10^x$, using 16 inches for the unit of x and 2 inches for that of y. From this read off $\log_{10} 1{\cdot}5$ and $\log_{10} 2$.

76. Find by logarithms $\frac{669 \times 37}{782}$ correct to 2 decimal places.

77. Prove that $\log_y\left(\frac{x}{y^z}\right) = \log_y x - z$.

78. If $x = \log_b a$, what is the log of a^n to the base b^n?

79. By the aid of tables, prove that approximately

$$200^{\frac{1}{10}} + \log_{10} {\cdot}2003 = 1.$$

80. Compute by tables $\frac{25}{7}\sqrt[3]{34{\cdot}3} \div \sqrt{14{\cdot}4}$.

81. If $\log \frac{1025}{1024} = \alpha$, and $\log 2 = \beta$, show that $\log 4100 = \alpha + 12\beta$.

82. Prove that $\log 1250 + \log {\cdot}343 - \log 49 + \log 25 + \log 6{\cdot}4 = 2 + \log 14$.

83. Find x correct to 2 decimal places if $5^x . 7^{x-1} = 11^{x+2}$.

84. Simplify $\log {\cdot}06 + \log({\cdot}6)^2 - \log 4 - \log 54 + 4$.

85. If the logarithm of a given number to base 4 is ${\cdot}3518$, what is its logarithm to the base 8?

86. Given $5^{x+2} = 8^{2x-1}$, find x correct to 3 decimal places.

87. Calculate the following by logarithms, and show how you would roughly check your results:

(i) pr^n, where $p = 93{\cdot}75$, $r = 1{\cdot}03$, $n = 4$;

(ii) $\frac{4}{3}\pi r^3$, where $\pi = \frac{355}{113}$, $r = 5{\cdot}875$.

88. The weight in pounds of a foot of iron piping is given by the formula $\frac{\pi t w}{144}(b + t)$, when the thickness is t inches, the bore is b inches, and $\pi = 3{\cdot}142$, w being the weight of a cubic foot of the iron. With $b = 3$, $t = \frac{3}{8}$, the weight of a foot is found to be $13{\cdot}3$ lb.; find w. Hence calculate, to the nearest pound, the weight of a foot if $b = 2\frac{1}{2}$, $t = \frac{1}{4}$.

89. The resistance to an express train on the level is given by the formula

$$R = 2{\cdot}5 + \frac{v^{\frac{5}{3}}}{50{\cdot}8 + 0{\cdot}0278L},$$

where R = resistance in pounds per ton drawn; v = velocity of train in miles per hour; L = length of train in feet.

Calculate R for a train 300 feet long when v is 60.

90. Make a list of the integral powers of 2 from index -2 to index $+4$. Use your values to draw a curve, the ordinates of which show the logarithms of numbers between 0·3 and 16·0 to base 2, using 1 cm. as unit for the numbers, and 5 cm. as unit for the logarithms. From your curve determine as accurately as you can $\log_2 1{\cdot}7$ and $\log_2 5{\cdot}4$.

From Tables of Logarithms, where logarithms of five-figure numbers are given to 7 places of decimals, we find that

$$\log 73603 = 4{\cdot}8668955, \text{ and } \log 73604 = 4{\cdot}8669014.$$

91. Calculate log 73603·2 correct to 7 decimal places.

92. .... log 7360·37 ... .

93. .... log ·0736036 ....

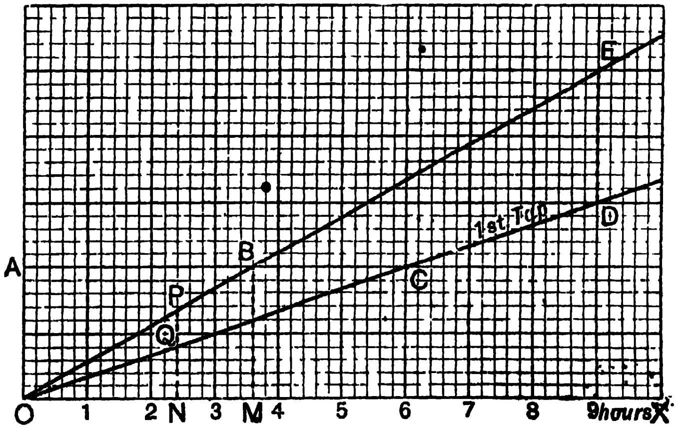
94. Find the number whose logarithm is 2·8668979.

95. .... 0·8669002.

*CHAPTER XL.

HARDER GRAPHICAL AND MISCELLANEOUS PROBLEMS.

256. *One tap will fill a cistern in 6 hours; a second will fill it in 9 hours: how long will they take to fill the cistern, running together?*



Measure time across the page along OX as shown in the diagram. [One-half in. (reduced) = 1 hour.]

Let OA (1 in.) measured up the page along OY denote the capacity of the cistern. OC is the graph of the first tap.

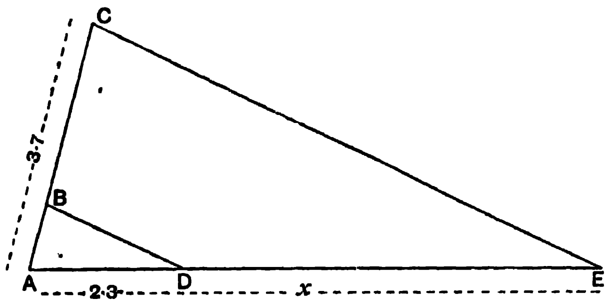
From D, the pt. where this meets the vertical 9 hour line, take DE = 1 in. upwards.

OE is the graph of the work done by the two taps.

Thus PN is the portion of the cistern filled by the two taps, running together, in time ON.

BM = OA. $\therefore$ OM is the reqd. time = 3·6 hours.

257. *Multiply* 3·7 *by* 2·3 *by a graphical method.*



Draw two str. lines AC, AE. With a convenient unit, make AB = 1, AC = 3·7, AD = 2·3.

Join BD and draw CE ∥ to BD.

Then $\frac{AE}{AD} = \frac{AC}{AB}$, *i.e.* AE × unity = AD × AC

$$= 2·3 \times 3·7.$$

$\therefore$ the number of units of length in AE represents the product 2·3 × 3·7.

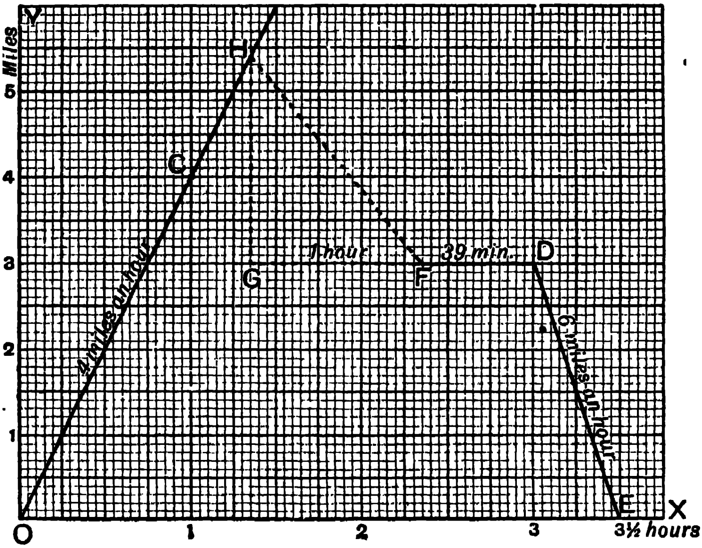
By measurement, AE = 8·51, the reqd. product.

Squared paper might be used for the above, BAD being made a rt. angle.

Division may be performed in a similar manner.

258. *A man walked from* A *to* B *at the rate of* 4 *miles an hour, without delay ran back at the rate of* 6 *miles an hour for half an hour, then waited for* 39 *minutes, and then completed the journey home, by*

walking the rest of the way, in an hour. How far is it from A *to* B, *if the whole journey occupied* $3\frac{1}{2}$ *hours, and what distance did he walk in the last part of the journey?*



[In the printed diagram, which is reduced, the side of each small square denotes one-tenth of an inch.]

Measure time along OX, taking 2 in. to denote an hour, and distance along OY, taking 1 in. to represent a mile.

OCH is the graph of 4 m. an hour, DE is the graph of the distance he runs at 6 m. an hour, DF = 39 minutes, the time he waits.

FG = 1 hour. Let the line through G parallel to OY meet OC at H.

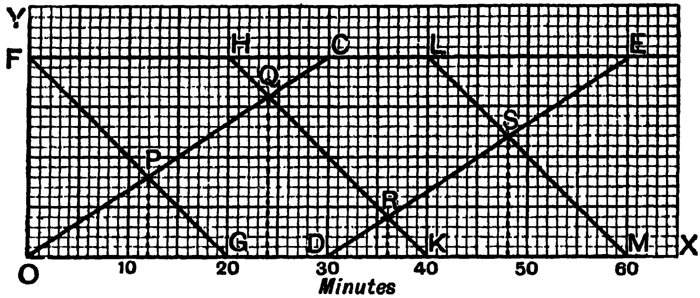
Then HF is the graph of his walk at the end of the journey.

Thus we see that the distance of H from OX = the distance of A from B. ∴ from the diagram, the dist. reqd. = 5·4 miles.

Also GH = 2·4 miles = the dist. he walked at the end of the journey.

N.B.—The order in which the different parts of the journey and the waiting are taken is immaterial.

259. A *walks at the rate of* 2 *miles an hour,* B *at the rate of* 3 *miles an hour, round a circular track* 1 *mile long, starting at the same point, and at the same time in opposite directions. Find the times of their first four meetings.*



The point to observe is, that between each two meetings the men together walk one mile.

Measure times along OX as shown in the diagram.

Take OF along OY to denote one mile.

OC is A's graph for the first 30 minutes (1 mile).

A is then at the starting point. ∴ we may take DE for his graph in the next 30 min.

Taking F as B's starting point, FG is his graph for the first 20 min., and as with A, HK his graph for the second 20 min. and LM his graph for the third 20 min.

The points P, Q, R, S, where these meet, give us the times reqd., which are 12, 24, 36, 48 minutes from the start.

260. *A man receives* 4 *shillings for every day that he works, but is fined* 1*s.* 6*d. for every day he is absent. After* 25 *days he receives* £2. 16*s. in wages. How many days was he absent?*

Take OA to represent 25 days, OB to represent 100 shillings, so that OC is the graph of the money the man earns.

Draw AD, the graph of his fines, from the point A, instead of from O.

Examining any ordinate PQN, we see that

PN represents the money he earns in ON days,

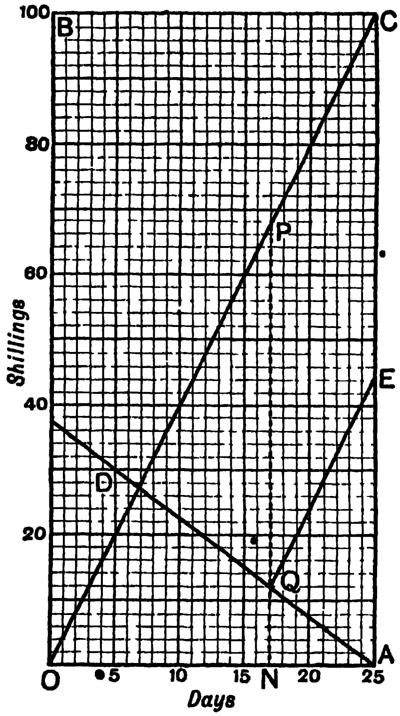
and QN .... amount of his fines in AN days;

∴ PQ represents the money he actually receives.

To solve this problem we have to draw PQ equal to 56 shillings.

Take CE equal to 56 shillings, and draw EQ parallel to OC to meet AD at Q. Draw the ordinate PQN.

$$PQ = EC = 56;$$



∴ ON represents the number of days he is at work,
and AN the number of days he is absent.

From the diagram we see that he is absent from work for 8 days.

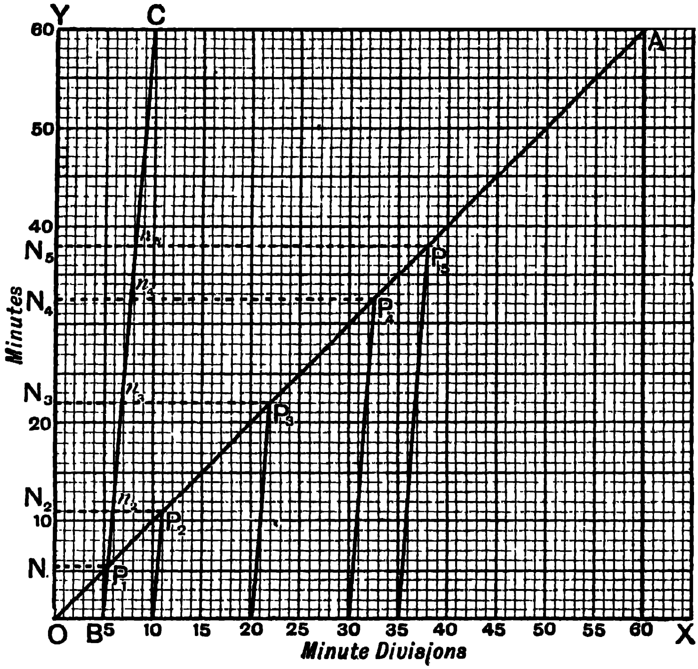
261. *Find graphically (to the nearest minute) the times after* 1 *o'clock when the hands of a clock are* (1) 5 *minute divisions apart,* (2) *first at right angles,* (3) 25 *minute divisions apart,* (4) *pointing in opposite directions.*

Measure the minute divisions across the page and minutes of

time up the page, as shown in the diagram. OA is the graph of the minute hand.

Remembering that the hour hand starts at 1 o'clock, and turns through 5 minute divisions in an hour, BC is its graph.

Drawing through the 10, 20, 30, and 35 minute division points straight lines parallel to BC, we obtain the points P_2, P_3, P_4, P_5.



Through these points draw parallels to OX.

$P_2n_2 = 5$ minute divisions. $\therefore$ ON_2 gives the time when the hands are 5 minute divisions apart.

Similarly

ON_3 gives the time when the hands are first at rt. angles,

ON_4 .... 25 min. divisions apart,

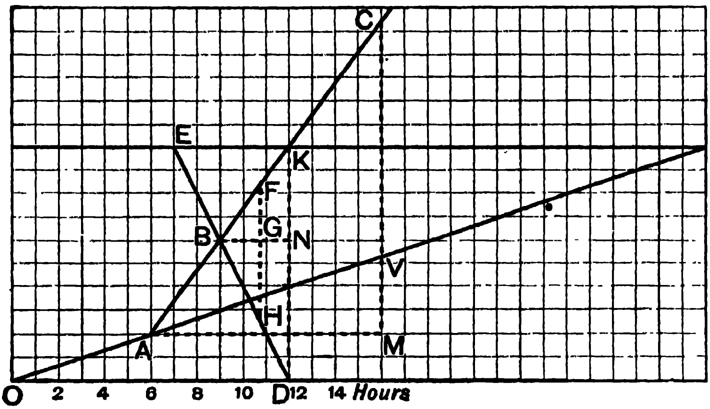
ON_5 ... . pointing in opp. directions.

The times, to the nearest minute, are

(1) 11 min. past one. (2) 22 min. past one.

(3) 33 .... (4) 38 ....

262. *A tap, which would fill a cistern in 30 hours, is set running; after 6 hours, an additional tap, which running alone would fill the cistern in 10 hours, is opened. After another 3 hours, a valve is opened and with the two taps still open, the cistern is emptied in 3 hours more. How long would the valve take to empty the full cistern if the taps were shut?*



Measure the hours across the page as shown in the diagram, and let OP, drawn up the page, represent the capacity of the cistern.

OA is the graph of the first tap.

From A, where it cuts the 6 hour line, measure AM equal to 10 hours, and draw the ordinate MVC meeting OA at V.

Take VC equal to OP, the capacity of the cistern, and join AC. AC is the graph of the work done by the two taps running together. Take the point B on this graph, 3 hours from A.

BD is the graph of the work done by the two taps and the valve.

Examining any ordinate FGH between BC and BD, we see that GF = the part of the cistern filled by the taps in time BG, and therefore FH = the part of the cistern emptied by the valve in the same time.

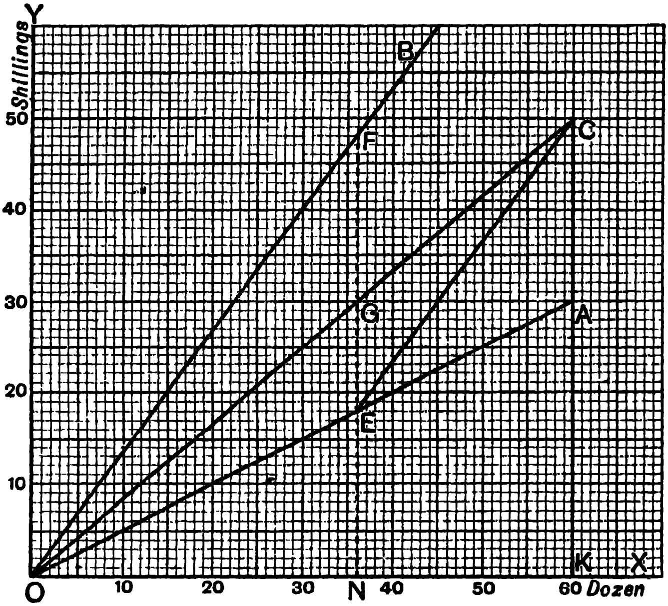
Hence to find the time taken by the valve alone to empty the full cistern, we have to find the points where the distance parallel to OP between BC and BD is equal to OP the capacity of the cistern.

KD is this line. Therefore BN denotes the time taken by the valve alone to empty the full cistern.

From the figure $\qquad BN = 3$ hours.

This should be verified by the algebraic method.

263. *A man mixes wine at* 30*s. a dozen with wine at* 80*s. a dozen. How many dozen of each kind must he take in order that a mixture of* 60 *dozen may be worth* 50*s. a dozen?*



In the diagram, OA is the graph of the 30*s.* wine, OB of the 80*s.* wine, OC of the mixture.

From C, the pt. (60, 50), draw CE parallel to OB to meet OA at E.

Draw the ordinate FGEN as shown.

ON $(= 36)$ gives the no. of dozen of the cheaper kind.

NK $(= 24)$ .... dearer kind.

Proof. The gain on ON dozen of the cheaper is CA × ON, *i.e.* GE × 60.

∴ if he sold ON dozen of the cheaper wine at 50*s.*, GE × 60 would represent his gain.

If he sold ON dozen of the dearer wine at 50*s*. FG × 60 would represent his loss.

Now $$\frac{ON}{NK}=\frac{OG}{CG} \text{ (similar } \triangle\text{s ONG, OKC)}$$
$$=\frac{FG}{GE} \text{ (........... OFG, CGE)};$$
$$\therefore\ GE=FG\times\frac{NK}{ON};$$

i.e. his gain on ON dozen of the cheaper = his loss on NK dozen of the dearer wine, when he sells the mixture at 50*s*.

∴ he sells 36 dozen of the cheaper with 24 dozen of the dearer.

Examples. XL. a.

The following problems should be solved by graphical methods. They may be verified algebraically.

1. One tap will fill a cistern in 6 hours, and a second will fill it in 4 hours. How long will they take to fill the cistern running together?

2. A tap would fill a cistern in 10 hours. When the cistern is half full a second tap is turned on and the cistern is full in one hour more. How long would the second tap, running alone, take to fill the cistern?

3. A man can dig over a garden in 5 days, and a boy in 10 days. What fraction of the garden do they dig over working together for 2 days?

4. A man starts digging a garden at a rate which would complete the work in 10 days. After working for 4 days, he takes 2 days' rest, and after that, working at a faster rate still does the work in the 10 days. Find approximately how long he would take to do all the garden working at the faster rate.

5. A does in 5 days a piece of work which B does in 7 days. For how many complete days must they be engaged to do the work when they work together?

6. A cistern has 3 pipes A, B, C; A and B can fill it in 4 and 5 hours respectively, and C can empty it in 2 hours. If these pipes are opened in order at noon, 1 o'clock, and 2 o'clock, find when the cistern will be empty.

7. A walks round a circular track one mile long in 20 minutes, and B motors round it in 5 minutes in the opposite direction, but starting from the same point. Draw graphs to shew when and where they meet, distances to be measured in A's direction.

8. A travels at the rate of 7 miles an hour, and B at 2 miles an hour round a circular track one mile long, starting at the same time from the same point in the same direction. Find the first three times when A passes B. Also write down the distances from the start, measured in the direction of the travellers, at which the passing points lie.

9. 1000 German marks are equal to £49. Find graphically the value of 550 marks in pounds, and £65 in marks, to the nearest pound, and 10 marks respectively.

10. Find the square root of 2·3 × 3·5.

11. In a hundred yards race, A beats B by 11 yards, and C by 19 yards. Find to the nearest yard how much B beats C by in 100 yards. (Use millimetre paper.)

12. A walks at the rate of 4 miles an hour, resting for half an hour at the end of each hour. B, starting 3 hours later, and travelling uniformly without resting, catches him up 14 miles from home. Draw their graphs, and determine B's pace.

13. If I buy oranges at the rate of 5 for 3*d.*, how much, to the nearest penny, shall I give for 57?

14. Two trains start at the same time, one from Liverpool to Manchester, and the other from Manchester to Liverpool, and running steadily complete the journey in 42 minutes and 56 minutes respectively. How long is it from the moment of starting before they meet?

15. A and B run over a course, B having 50 yards start. A runs 4 yards while B runs 3 yards, and arrives at the end of the course 150 yards ahead of B. Find the length of the course.

16. Two men run at uniform rates over a course of 4000 yards. One starts 24 seconds after the other and arrives 16 seconds before him. Where does he pass him?

17. Suppose that it takes a train just 6 days to run the whole length of the road, and that one train leaves each end of the road each morning. How many trains will a person meet going the length of the road, not counting the train which arrives just as he starts, nor the train which starts just as he arrives?

18. A man receives 3*s.* 6*d.* for every day that he works, but is fined 1*s.* for every day he is absent. After 20 days he receives the same wages that he would have earned by working steadily for 11 days. How many days was he absent from work?

19. A, walking from P to Q at the rate of 4 miles an hour, starts one hour before a coach travelling at the rate of 12 miles an hour, and is picked up by the coach. On arriving at Q he finds that his coach journey has lasted 2 hours. Find the distance from P to Q.

20. A, B and C travel from the same place at the rates of 4, 5 and 6 miles an hour respectively, and B starts 2 hours after A. How long after B must C start in order that they may overtake A at the same instant?

21. A man walked from A to B at the rate of 3 miles an hour, bicycled back without delay for 2 miles at the rate of 8 miles an hour, and walked the remaining distance home at the rate of 2 miles an hour, taking 4 hours over the journey. How far is it from A to B?

22. A man walks from A to B at the rate of 4 miles an hour, waits half an hour at B, and then returns at the rate of 3 miles an hour. He took $3\frac{1}{2}$ hours over the journey. Find the distance from A to B.

23. A man did a journey of 11 miles in the following manner. He walked part of the way, then bicycled at the rate of 11 miles an hour, and finally completed the journey by walking for 15 minutes. If his walking was at the rate of 3 miles an hour, and he took 3 hours over the journey, find how far he bicycled.

24. A man walked a certain distance at the rate of 4 miles an hour, and then ran part of the way back at the rate of 6 miles an hour, walking the remaining distance home in 15 minutes. The whole journey took him

1 hour and 20 minutes. How far did he run, and what is the distance? If a second man does the reverse double journey uniformly, starting at the same time, and occupying the same time, when and where will they meet?

25. Two towns are 50 miles apart: A is to leave one of these towns at 6 o'clock and to arrive at the other at noon, making four stops of half an hour each at 10, 20, 30 and 40 miles from the starting point. B leaves the other end of the road at 7 o'clock, travels 20 miles an hour for an hour, then turns back and retraces his course for an hour at the rate of 10 miles an hour, then turns round and advances again at such a rate as to meet A as he is starting from his third halt: continuing at the same rate B meets at 10.30 a third man C, who left the first end of the route 2 hours later than A, and has been going at a uniform rate. At what rate has C been travelling, and where did B meet him?

26. A travels 6 miles in the first hour, 5 in the second, 4 in the third, and so on. B starting 14 miles behind him travels at the uniform rate of 7 miles an hour. When and where does he catch B up?

27. A tap which would fill a cistern in 3 hours, and a plug which would empty it in 7 hours, are both opened at the same instant, when the cistern is empty. How long will they take to fill the cistern?

28. A tap and a plug when both open fill an empty cistern in 120 minutes, whilst the tap alone would fill it in 40 minutes. How long would it take the plug to empty the full cistern?

29. Find graphically (to the nearest minute) the times when the hands of a clock are (1) coincident in direction, (2) at right angles, (3) 20 minute divisions apart between the times given below:

(i) 2 o'clock and 3 o'clock. (ii) 6 o'clock and 7 o'clock.

(iii) 9 o'clock and 10 o'clock. (iv) 11 o'clock and 12 o'clock.

30. If 4 men, or 5 women, or 6 boys can do the same amount of work in the same time, how long will one man, one woman, and one boy take to do a piece of work which 4 men do in $2\frac{1}{2}$ days?

31. A, B, and C together do one-quarter of a piece of work in one day, B and C do one-half in 4 days, C completes it in 6 days more. How long would A and B take to do it, working alone?

32. Two plugs are opened in a cistern containing 192 gallons of water: after 3 hours one of the plugs becomes stopped, and the cistern is emptied by the other in 11 more hours. Had 6 hours elapsed before the stoppage, it would have required only 6 hours more to have emptied the cistern. How many gallons will each hole discharge in an hour, supposing the discharge uniform?

33. Two workmen of unequal efficiency can make a drain in 60 days. After working for 20 days, A falls ill, and his place is taken by C, whose efficiency is the same as that of B, and they get the drain finished in 80 days. In how many days could A or B alone have made the drain?

34. A man mixes tea at 1*s.* 1*d.* per lb. with tea at 1*s.* 11*d.* per lb., and sells the mixture, without gain or loss, at 1*s.* 4*d.* per lb.: find the percentages of the two kinds in the mixture.

35. A man mixes tea at 1*s.* per lb. with some at 2*s.* 3*d.* per lb., and sells the mixture at 1*s.* 4*d.* per lb., without gain or loss. How many lbs. of each kind did he sell?

36. Two taps running together would fill a cistern in 15 hours. After running together for 6 hours, one is turned off, and the other fills the cistern in 30 hours more. In what time could each fill the cistern, running separately?

37. A tap which would fill a cistern in 4 hours is opened. After it has been running an hour, a second tap is opened which would, running alone, fill the cistern in 8 hours. At the end of a second hour, a plug is opened, and the cistern which was empty at the start, is empty again in 7 hours from the start. How long would the plug, running alone, take to empty the full cistern?

38. A cyclist rides 3 miles an hour faster downhill than uphill, and takes the same time to ride 22 miles downhill and 48 uphill that he takes to ride 50 miles downhill and 27 miles uphill. What are his speeds uphill and downhill?

39. The population of a town increases uniformly, and in each period of 3 years the increase is 20 per cent. of the population at the beginning of that period. If the population was 90,000 in January 1903, what will it be in January 1906, and 1912, and what was it in January 1897?

By drawing a graph, find the population approximately in January of each year from 1900 to 1905.

40. A and B are points 20 miles apart. At noon one man starts from A to walk to B at the rate of 4 miles an hour, and at 2 p.m. another man starts after him on a bicycle at 10 miles an hour. Draw a diagram on ruled paper to show how far they are apart at any given time, and at what times they pass any given point between A and B.

[Scale to be 5 miles = 1 inch, and 1 hour = 1 inch.]

Also find from the diagram when and where the cyclist overtakes the man walking.

41. A and B are at two places P and Q 30 miles apart. They start at the same instant to travel from P to Q and Q to P respectively and meet at R, 12 miles from P. If A arrives at Q $2\frac{1}{2}$ hours after B arrives at P, find graphically their rates of travelling.

42. A tap A fills one-half of a cistern in the same time that a tap B fills three-quarters of an equal cistern. Starting at the same time B fills the one cistern 2 hours sooner than A fills the other. Find how long each tap takes to fill its cistern.

43. Two men, starting at the same time from two places A and B, travel from A to B and B to A respectively at uniform rates. They are 30 miles apart in 3 hours and meet in 9 hours at a point 30 miles from A. How far is it from A to B, and what are their rates of travelling?

44. Two men, walking from A to B and B to A respectively, and starting at the same time, meet in 7 hours at a point $26\frac{1}{4}$ miles from A, and are 15 miles apart in 4 hours. How far is it from A to B, and what are their rates of travelling?

45. A man, who can row 2 miles against a stream in the same time as 3 miles with it, rows 15 miles and back in $6\frac{1}{4}$ hours. Find his times of rowing up and down.

46. Two pipes together fill a cistern in $2\frac{6}{7}$ hours. Running singly, one takes 6 hours longer than the other to fill it. How long does each tap take to fill it?

47. A and B start together from the foot of a mountain to go to the summit. A would reach the summit half-an-hour before B, but missing

his way, goes a mile and back needlessly, during which he walks at twice his former pace. He does the rest of the walk at his original pace, and reaches the summit 6 minutes before B. C starting 20 minutes after A and B, and walking at the rate of $2\frac{1}{7}$ miles per hour, reaches the summit 10 minutes after B. Find the rates of walking of A and B, and the distance from the foot to the summit of the mountain.

Draw the graphs of:

48. $xy=0$. **49.** $x^2+y^2=0$. **50.** $(x-5)(y-6)=0$.

51. $(x-5)^2+(y-6)^2=0$. **52.** $y=\frac{1}{x-1}$. **53.** $y=(x-1)(x-2)$.

Examples. XL. b.

MISCELLANEOUS PROBLEMS.

1. A and B run a race; B has 50 yards start, but A runs 20 yards while B runs 19; what must be the length of the course, that A may come in a yard ahead of B?

2. A man buys a certain quantity of apples to divide among his five children in succession. To the eldest he gives half the whole, all but 8 apples; to the second he gives half the remainder, all but 8 apples; to the third half the second remainder, all but 8 apples; to the fourth half the third remainder all but 8 apples. To the fifth he gives the 20 apples which remain. Find how many he bought.

3. One army contains 3 men for every 2 in another army. In a battle the first army loses 1600 men and the second 600 men. After this there are only 4 men in the first army for every 3 in the other. How many men were there in each army at first?

4. A trader begins business with a certain capital. The first year he spends £100 on himself, and at the end of the year finds that the rest of his capital has increased by one-third. At the end of the second year, after again spending £100 on himself, he finds that the capital with which he began the second year has (in a similar manner) increased by one-third, and that he now has £1466. 13*s*. 4*d*. How much had he to begin with?

5. Find two integers whose sum shall be 21, and which shall be such that the sum of the digits of one of the integers shall be double the other integer.

6. Find what quantity of tea at 2*s*. per lb. must be mixed with 30 lb. of tea at 3*s*. per lb. in order that 2*s*. may be gained by selling 10 lbs. of the mixture at 2*s*. 6*d*. per lb.

7. A man sets out to walk from A to B, a distance of 16 miles, at the rate of $3\frac{1}{2}$ miles an hour. Three-quarters of an hour after, a man sets out from B to meet him, at the rate of 4 miles an hour. When and where will they meet?

8. A and B possess together £22. After A has given to B one-fourth of his money, then one-third of what remained to him, and then £4 more, he has left only one-tenth of what B now has. How much money had each originally?

9. In a bicycle race A goes x feet per second and B goes $\frac{7x}{11}$ miles per hour. After 3 minutes A is 88 yards ahead of B: find x.

10. There is a number from which two other numbers are formed, one by adding 4 to it and the other by subtracting 4 from it; and the difference of the squares of these numbers is 144. What is the number?

11. A dealer buys $2a+b$ sheep at c shillings each, and sells $a+b$ of them at a profit of d shillings each, disposing of the remainder at a loss of d shillings each: find his gain by the whole transaction.

12. Of a swarm of bees clustered on a tree, the square root of half their number flew away. Eight-ninths of the original number then departed, leaving but two behind. How many were there at first?

13. A town is supplied with water from a reservoir into which the water has first to be pumped. The pumps if worked constantly could, in the case of no leakage or consumption of water, fill the reservoir in x days. They, however, work only on week-days, and for only 12 hours a day. Also the town consumes a reservoir-full in $3x$ days, using it night and day, and all days, and by leakage a reservoir-full is lost in $39x$ days. On Monday morning when the pumping begins the reservoir is empty. Find x, that on the next Saturday evening but two, when the pumping leaves off, the reservoir may be for the first time full.

14. A clock is set going at 12 o'clock, but loses four minutes an hour; at what true time between 2 and 3 o'clock will its hands point in opposite directions?

15. A grocer buys a number of eggs at 6*s*. 6*d*. per hundred. He sells all but 69 of them at the rate of 11 for a shilling, and finds that he has received 30*s*. more than he gave for the whole number. How many eggs did he buy?

16. There are two mixtures of wine and water, one contains twice as much water as wine, and the other three times as much wine as water. How much must be taken from each mixture to fill a pint vessel in which the wine and water are equally mixed?

17. A started six minutes before B and walked nine times round a circular path, finishing just as B completed his seventh round. A now rested for five minutes, while B went on with speed increased to half as much again as before, and just as he completed his twelfth round A, who had resumed his walk with double speed, finished his sixteenth round. How long do A and B require at first to walk round a path?

18. A litre of water weighs a kilogram, and a litre of another liquid weighs 1·34 kilograms. A mixture of the two weighs 1·27 kilograms per litre. Determine the volume of each in a litre of the mixture.

19. Potatoes are sold so as to gain 25 per cent. at 6 lbs. for 5*d*.: find the gain per cent. when they are sold at 5 lbs. for 6*d*.

20. A grocer gains 20 per cent. by selling at 2*s*. a lb. a mixture formed by mixing with 7 lbs. of a common tea 2 lbs. of a better kind. But if he had mixed 7 lbs. of the latter with 2 lbs. of the former kind, he would have lost 20 per cent. by selling the mixture at that price. What did each kind of tea cost him per lb.?

21. A man has three sons, born at equal intervals of time, whose united ages are at present equal to his own age. Show that when the age of the middle one is double of what it is now he will be exactly half his father's age.

22. A cricketer had a certain number of innings during the season; he finds that if he had not played the last, when he only made 5 runs, or if he had played an extra innings and made 31 runs, he would in either case have increased his average by 1. How many innings did he have, and what was his average?

23. Two purses contain money in such proportion that for every £1 there is in one of them there is 15*s*. 6*d*. in the other; and if £1 be taken out of the latter and put into the former, the difference of the contents will then be £3. 2*s*. 6*d*.; how much money is there in each purse?

24. I take a party by rail. Three times the number of those in my party who require whole tickets is equal to twice the number of those who require half-tickets. Find the number in my party, if 14 tickets suffice for all.

25. In a division in parliament, the Government had a majority of 18, and it was found that 53 per cent. of the whole number had voted for the Government. Find how many voted on each side.

26. The age of a father is twice the sum of the ages of his two sons, the elder of whom is twice as old as the younger. Nine years hence the father will be three times as old as his younger son. Find their ages.

27. I spend half of a certain sum of money in buying wheat at 28*s*. a quarter, and the other half in buying barley at 24*s*. a quarter. If the wheat had cost 32*s*. 3*d*., and the barley 26*s*. 8*d*. a quarter, I should have altogether spent £13. 5*s*. more. How many quarters of each did I buy?

28. On a certain journey I walked at $3\frac{1}{2}$ miles an hour, until the number of miles left to be walked was three less than the number of hours I had been walking. Then I walked the rest of the way at 4 miles an hour. If the whole journey occupied 10 hours 55 minutes, how many miles long was it?

CHAPTER XLI.

PAPERS FOR REVISION.

XLI. a.

1. Write down the square of $x+x^{-1}-1$, and the cube of $a^{\frac{1}{3}}+b^{\frac{1}{3}}$.

2. Find the value to 2 decimal places of $\dfrac{15+\sqrt{10}}{15-\sqrt{10}}+\dfrac{30-\sqrt{10}}{15+\sqrt{10}}$.

3. If $a:b=c:d$, then $a+\dfrac{1}{b}:b+\dfrac{1}{a}=c+\dfrac{1}{d}:d+\dfrac{1}{c}$.

4. If $A \propto BC$, and $B^2 \propto AC$, prove that $A \propto B^{\frac{3}{2}} \propto C^3$.

5. A can run round a circular path 5 times while B can run round another 6 times. If their rates are as 9 to 7, compare the lengths of the paths and determine how much per cent. A must increase his speed to run round his path 6 times while B with unaltered speed runs round his 5 times.

6. Find by logarithms the cube root of 4913.

7. Solve $x^2-3\sqrt{2x^2-x-2}=x(1-x)$.

XLI. b.

1. If $a:b=c:d$, prove $\dfrac{a^2}{b^2}=\dfrac{2a+c}{2b+d}\times\dfrac{2a-c}{2b-d}$.

2. Find the cube root of 3 to 3 decimal places.

3. The distance fallen by a body varies as the square of the time occupied, and it falls 16 feet in the 1st second. Find how far it falls in 6 seconds, and how far in the 6th second.

4. Prove that ax^3+bx^2+cx+d is divisible by $x-1$ if $a+b+c+d=0$.

5. Simplify $2\sqrt{72}+7\sqrt{8}-\sqrt{98}+\frac{6}{\sqrt{2}}$.

6. If $M=PR^n$, find the nearest integral value of n, when $M=2P$ and $R=1\cdot05$.

7. How much are eggs a score when a rise of 20 per cent. in the price would make a difference of 80 in the number for a sovereign?

XLI. c.

1. Solve $\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=3$.

2. Find the sides of a right-angled △ whose area is 30 sq. in. and hypotenuse 13 in.

3. Divide 35 by $2\sqrt{6}-\sqrt{3}$.

4. Draw the graph of $y=x^2-4x$, and hence solve the equation $x^2-4x=3$.

5. If $a:b=c:d$, then $\frac{a^{2n}+b^{2n}+c^{2n}+d^{2n}}{a^{-2n}+b^{-2n}+c^{-2n}+d^{-2n}}=(abcd)^n$.

6. In which is the greater waste, in cutting a square out of a circle or a circle out of a square? Find the ratio of the waste to the original area in each case.

7. Given that the area of a circle varies as the square of the radius, prove that the area of a circle of radius 13 inches = the sum of the areas of two circles of radius 5 and 12 inches.

XLI. d.

1. Express $(a^2+b^2)(c^2+d^2)$ as the sum of two squares.

2. Solve $x^2-6x+16=4\sqrt{x^2-6x+12}$.

3. Draw the graph of $y=10^x$ from $x=-1$ to $x=1\cdot5$, and find from it the common logarithms of $1\cdot25$, 23, and $6\cdot3$.

4. There are two vessels A and B each containing a mixture of water and wine, A in the ratio of 2 to 3, B in the ratio of 3 to 7. What quantities must be taken from them to form a third mixture which shall contain 5 gallons of water and 11 of wine?

5. Show graphically that the ratio $a:b$ lies in value between the ratio $na:nb+p$ and the ratio $na:nb-p$.

6. Simplify $2^{-1}.3^{\frac{1}{2}}.6^{\frac{3}{2}}+9^{-\frac{1}{4}}.12^{\frac{3}{2}}.2^{\frac{1}{6}}+2^{-\frac{3}{2}}$.

7. The volume of a cube is $17\cdot5$ cubic feet; find by logarithms the length of its edge.

XLI. e.

1. Simplify $\sqrt{\dfrac{22+6\sqrt{5}}{8-3\sqrt{5}}}$ and find its value to 3 decimal places.

2. A boy runs a mile race in $6\frac{1}{4}$ minutes; he does the last lap, one-quarter of a mile, at 3 miles per hour faster than he does the first three laps. Find his pace for the last lap.

3. Solve $\sqrt{2x+8}-2\sqrt{x+5}+2=0$.

4. If $a:b=c:d$, show by similar triangles that $\dfrac{a^2+b^2}{b^2}=\dfrac{c^2+d^2}{d^2}$.

5. The area of a right-angled $\triangle$ is 84 sq. in., and its perimeter 56 in. Find the sides.

6. The volume of a sphere varies as the cube of its radius. Prove that three spheres of radii 3, 4, 5 are together equal in volume to one of radius 6.

7. If $3^x=9\times 3^{\frac{1}{x}}$, find x to 2 decimal places.

XLI. f.

1. The third proportional to two numbers is 162, and the mean proportional 6. Find the numbers.

2. A man receives a fixed allowance of £75 a year, but increases his expenditure by 25 per cent. every year. At the end of 4 years he is £69 in debt. How much did he save the first year? *Represent this graphically.*

3. Which is the greater, $3:4$ or $3x^2+6x+1 : 4x^2+8x+1$?

4. If $\dfrac{a}{b}=\dfrac{c}{d}$, $a:d=(a+b)(a-c):(b-d)(c+d)$.

5. Prove $\log_a N=\log_a b \,.\, \log_b N$; and find $\log\sqrt{5}$ to the base $\cdot 008$.

6. Three diamond rings weigh 141, 116, 79 grains, and it is known that the weights of the gems are 3, 2, 1 carats. The 1st ring is worth £82, the 2nd £36. 16s. What is the value of the 3rd? [1 carat=4 grains, and the value of a diamond varies as the square of the weight.]

7. Find the value of $\sqrt{1\cdot 5+\sqrt{2}}$ graphically and algebraically.

XLI. g.

1. Find a number which added to each of the numbers 6, 8, 10, 13 will make them proportionals.

2. Find the value of $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$ when $x=\dfrac{2a}{1+a^2}$.

3. Find the number whose logarithm is $\frac{1}{3}$ of $\bar{2}\cdot 8471$.

4. If a, b, c are in continued proportion, $a+2b+c=\dfrac{(b+c)^2}{c}$, and

$$\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}=\frac{a^3+b^3+c^3}{a^2b^2c^2}.$$

5. Given that $w\propto x+y$ and $y\propto z^2$, and when $z=2$, $w=26$, $x=1$, $y=12$; express w in terms of x and z.

6. Solve

$$\text{(i)}\quad \left.\begin{array}{l} x+\sqrt{x+y}=12-y \\ x^2+y^2=41 \end{array}\right\},$$

$$\text{(ii)}\quad 3^{2x}-2\times 3^{x+1}=567.$$

7. A walks from Cheltenham to Gloucester at 4 miles an hour: B starts at the same time from Gloucester and bicycles to Cheltenham and back at 10 miles an hour. Find *graphically* where B passes A, the distance between the two places being 8 miles.

XLI. h.

1. Express $(a^2+b^2)(c^2+d^2)(e^2+f^2)$ as the sum of two squares.

2. The height of a spiral staircase is 40 feet and the internal diameter 3·183 feet. The inner hand-rail makes 3 complete revolutions: find its length.

3. Solve $2x^3+7x^2+7x+2=0$.

4. Prove from the definition that $\log_y \dfrac{x}{yz}=\log_y x-1-\log_y z$.

5. If a, b, c, d are in continued proportion $(b+c)(b+d)=(a+c)(c+d)$, and $(a^2+b^2+c^2)(b^2+c^2+d^2)=(ab+bc+cd)^2$.

6. Find *geometrically* the value of $\sqrt{1+\sqrt{5}}$.

7. Find $\log(2\sqrt{7}\div 3\sqrt{11})^{\frac{1}{6}}$.

XLI. i.

1. Add together $\sqrt{20}$, $3\sqrt{125}$, $4\left(\dfrac{80}{9}\right)^{-\frac{1}{2}}$, $\dfrac{9}{\sqrt{5}}$, $\sqrt{\dfrac{324}{5}}$ and $\sqrt[4]{\dfrac{25}{16}}$.

2. Regarding x as a known quantity, solve the following quadratic equation for y: $\quad -2x^2+y^2+xy+11x-2y-15=0$.

Hence factorise the expression.

3. Find, by logarithms, the square root of 3·015 correct to 3 significant figures.

4. If $y\propto\dfrac{1}{x}$ and $x=2$ when $y=2$ draw the graph of the resulting equation, and thus find the value of y when $x=3{\cdot}14$.

5. Two cisterns connected by a pipe contain 25 and $24\frac{1}{2}$ gallons of water. How much must flow from the first into the other that their contents may be in the ratio of 5 to 6?

6. If a, b, c, d are proportional,

$$(a-b)(c-d)=\frac{b}{d}(c-d)^2=\frac{d}{b}(a-b)^2.$$

7. Solve

$$\text{(i)}\quad x^2+2\sqrt{x^2+2x+3}=12-2x,$$

$$\text{(ii)}\quad \frac{1}{x+\sqrt{2-x^2}}+\frac{1}{x-\sqrt{2-x^2}}=\frac{x}{2}.$$

XLI. k.

1. With the same axes trace the graphs of $y=x+\frac{2}{x}$ and $y=x^2$.

Note from your graphs for what value of x the value of $x+\frac{2}{x}$ is the same as the value of x^2.

2. Solve $$\sqrt{x}+\sqrt{a+x}=\frac{2x}{\sqrt{a+x}}.$$

3. A and B start to run a race to a certain post and back again. A returning meets B at 80 yards from the post and arrives at the starting point 1 minute before him. If he had then returned immediately to meet B, he would have met him when B had still to run $\frac{1}{6}$th of the distance between the post and the starting-place. Find the length of the course and the duration of the race.

4. α, β are the roots of $x^2+px+q=0$. Find $\alpha\beta^{-1}+\alpha^{-1}\beta$.

5. Extract the square root of $16+\sqrt{252}$.

6. Find the product of $a^{\frac{1}{3}}$, $a^{-\frac{3}{4}}$, $\sqrt[3]{a^4}$, $a^{\frac{1}{12}}$, $\sqrt[8]{a^{\frac{25}{3}}}$ and $\left(a^{-\frac{7}{4}}\right)^{\frac{7}{6}}$.

7. If x men do as much work as y women, and p women as much as q boys, what should be the ratio of a man's wages to a boy's?

XLI. l.

1. A man loses $\frac{1}{3}$ of his money; then wins £10; loses $\frac{1}{3}$ of what he then has, and wins £20, and finds that he has exactly what he had at starting. What had he?

2. Simplify $\dfrac{2+\sqrt{3}}{\sqrt{2}+\sqrt{2+\sqrt{3}}}+\dfrac{2-\sqrt{3}}{\sqrt{2}-\sqrt{2-\sqrt{3}}}$.

3. If $x\propto\frac{z}{y^2}$, and $z^2\propto\frac{y}{x}$, prove that $x\propto\frac{1}{y}\propto\frac{1}{z}$.

4. To do a piece of work A takes m times as long as B and C together, B n times as long as C and A together, C p times as long as A and B together. Prove $\frac{1}{m+1}+\frac{1}{n+1}+\frac{1}{p+1}=1$.

5. From the equations $x^2+y^2=25$, $xy=12$ find the ratio $x:y$.

6. Verify this by drawing the graphs of the two equations and finding the equations of the lines joining the origin to their intersections.

7. Find by logarithms the 4th root of $\frac{3546}{121}$ and verify by extracting the square root of the square root.

XLI. m.

1. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\sqrt{ab}+\sqrt{cd}+\sqrt{ef}=\sqrt{(a+c+e)(b+d+f)}$.

2. Simplify $\frac{\sqrt{5}+1}{\sqrt{5}-\sqrt{2}}+\frac{\sqrt{5}-1}{\sqrt{5}+\sqrt{2}}-\frac{2(\sqrt{5}-\sqrt{2})}{\sqrt{89-28\sqrt{10}}}$.

3. There are two vessels, one containing 7 gallons of water and 3 of spirit, the other 4 gallons of water and 6 of spirit. How much must be taken from each to fill a 10 gallon vessel with equal parts of water and spirit?

4. Find the mean proportional between $4\cdot\dot{6}$ and $10\cdot5$.

5. If $(h-a)(1-a)=(h-b)(1-b)=k$, find k in terms of a and b only.

6. In the equation $ax^2+bx+c=0$, $2b^2=9ac$, find the ratio of the roots.

7. The value of a silver coin varies as the product of the thickness and the square of the diameter. Two silver coins have their diameters in the ratio 5 : 4. Find the ratio of their thicknesses if the value of the 1st is twice that of the 2nd.

XLI. n.

1. By dividing a line in extreme and mean ratio, find by measurement the value of $\frac{1}{2}(\sqrt{5}-1)$.

2. If $a \propto b$ and c jointly, and $b \propto d^2$, and $c \propto$ the reciprocal of a, prove that $a \propto d$.

3. If log 75 lie between 2 and 3, what integer is the base?

4. How many figures are there in 5^{100}, given $\log 2 = \cdot30103$?

5. Solve $\sqrt{3x-5}+2\sqrt{x-2}=\sqrt{5x+1}$.

6. A father gave his son a certain sum of money, telling him that at the end of every year he would give him as much as he then had left. The son spent £100 a year, and at the end of 4 years had nothing left. How much did he receive at first?

7. A and B start to run a race. When A has run c yards, B is d yards behind. B then increases his pace in the ratio $m:n$. The race lasts t minutes longer and results in a dead heat. Find the distance and the rates of running.

XLI. p.

1. Solve the equation $\sqrt{7x}=\sqrt{3x+12}+\sqrt{3-x}$.

2. The perimeter of a right-angled triangle is 6 times the least side. Find the ratio of the sides containing the right angle.

3. Find $\log\sqrt[3]{(\frac{5}{12})}$.

If $\frac{\log x}{y-z}=\frac{\log y}{z-x}=\frac{\log z}{x-y}$, prove that $x^xy^yz^z=1$.

5. The square of the time during which a body will slide down a smooth incline varies as the square of the length and inversely as the height. If the time is 1 second when the height is 4 feet and length 8 feet, what must be the height of a plane 3 feet long that a body may slide down in half a second?

6. In a walking race A gives B a start of two-thirds of a mile. After A has walked for 36 minutes he overtakes B, but 18 minutes later he meets with an accident which detains him 27 minutes. On resuming the race, A walks for an hour and a half and again overtakes B two-thirds of a mile from the winning post. Find the length of the course.

7. If $a(by+cz-ax)=b(cz+ax-by)=c(ax+by-cz)$, then

$$\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a+b}.$$

XLI. q.

1. Find a quadratic function of x which shall have the values 1, 14, $\frac{23}{2}$ when $x=1$, 3, and -2 respectively.

2. Express as a proportion the statement

$$(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d),$$

and prove that $a:b=c:d$.

3. Two travellers start towards each other from two places, one walking a mile an hour faster than the other. They meet a mile from the middle of the distance, and find that the time of the faster over the whole distance would be $3\frac{1}{3}$ hours. Find the distance.

4. The weight of a spherical shell is $\frac{7}{8}$ of what it would be if it were solid. Compare the inner and outer radii, given that the volume varies as the cube of the radius.

5. In a proportion the sum of the means $=8$, sum of extremes $=16$, the sum of squares of all the terms $=260$. Find the proportion.

6. Express in decimals 10^{-1}, $10^{-\cdot 5}$, $10^{-\cdot 25}$, $10^{\cdot 25}$, $10^{\cdot 5}$, $10^{\cdot 75}$. Draw the graph of $y=10^x$ on a large scale for values of x between -1 and 1.

Find from the graph the values of $10^{\cdot 2}$, $10^{\cdot 3}$, $10^{-\cdot 73}$.

Check the results by means of a table of logarithms.

7. Find $3^{\frac{1}{8}}$ by extracting square roots, and state to how many places your result is correct.

CHAPTER XLII.

ARITHMETICAL PROGRESSION.

264. A series of quantities is said to be in **Arithmetical Progression** when each term is formed by adding a constant quantity, positive or negative, to the preceding term.

This constant quantity is called the **common difference.**

The common difference = the 2nd term − the 1st term
= the 3rd term − the 2nd term
= the 4th term − the 3rd term, and so on.

3, 5, 7, 9, ... form an A.P. whose common difference is 2.
20, 17, 14, 11, ... −3.
x, $5x$, $9x$, $13x$, ... $4x$.

265. *To find the nth term of an* A.P.

Let a = the 1st term, d = the common difference.

The 2nd term $= a + d$.

The 3rd term $= a + 2d$, etc.

$\therefore$ **the nth term** $= \mathbf{a + (n-1)d}$.

266. *To find the sum of n terms of an* A.P.

Let a = the 1st term, d = the common difference, l = the last term, s = the sum of n terms.

$$s = a + (a + d) + (a + 2d) + \dots + l.$$

Also s = the sum of the same terms in reverse order

$$= l + (l - d) + (l - 2d) + \dots + a.$$

By addition

$$2s = (a + l) + (a + l) + (a + l) + \dots = n(a + l).$$

$$\therefore\ \mathbf{s = \frac{n}{2}(a + l)}. \quad \dots\dots\dots\dots(1)$$

But $$l = a + (n-1)d. \quad \dots\dots\dots\dots(2)$$

$$\therefore\ s = \frac{n}{2}\{a + a + (n-1)d\}$$

$$= \mathbf{\frac{n}{2}\{2a + (n-1)d\}}. \quad \dots\dots\dots\dots(3)$$

267. If we insert between two quantities x and y a series of quantities which, including x and y, form an arithmetical progression, these quantities are called **Arithmetic Means.**

Example. 4, 6, 8, 10 are 4 arithmetic means between 2 and 12; for 2, 4, 6, 8, 10, 12 are in A.P.

268. *To insert n Arithmetic Means between x and y, i.e.* to form an A.P. of $n+2$ terms beginning with x and ending with y.

$$y = \text{the } (n+2)\text{th term} = x + (n+1)d.$$

$$\therefore \; d = \frac{y-x}{n+1}.$$

$$\therefore \text{ the means are } x + \frac{y-x}{n+1}, \quad x + \frac{2(y-x)}{n+1}, \quad x + \frac{3(y-x)}{n+1}, \text{ etc.}$$

269. *To find the Arithmetic Mean between x and y, i.e.* to insert one mean between x and y.

Let A be the required mean.

$$\therefore \; A - x = y - A.$$

$$\therefore \; A = \frac{x+y}{2}.$$

Example 1. The 5th term of an A.P. is -5, and the 11th term -23: find the 30th term and the sum of 30 terms.

Let the 1st term be a, the common difference d.

$$a + 4d = \text{the 5th term} = -5,$$

$$a + 10d = \text{the 11th term} = -23;$$

$$\therefore \; 6d = -18; \quad \therefore \; d = -3, \quad a = 7;$$

$$\therefore \text{ the 30th term} = 7 - 29 \times 3 = 7 - 87 = -80.$$

$$\text{The sum} = \frac{n}{2}\{a + l\} = \frac{30}{2}\{7 - 80\} = -15 \times 73 = -1095.$$

We might find the sum of 30 terms without finding the 30th term.

For $$s = \frac{n}{2}\{2a + (n-1)d\} = \frac{30}{2}\{14 - 87\} = -1095.$$

Example 2. Insert 5 arithmetic means between 11 and 37.

Here 11 and 37 with the intermediate terms form an A.P. of 7 terms.

The first term is 11, the 7th is 37.

i.e. $$11 + 6d = \text{the 7th term} = 37;$$

$$\therefore \; 6d = 26; \quad \therefore \; d = 4\tfrac{1}{3}.$$

$\therefore$ the required means are $15\frac{1}{3}$, $19\frac{2}{3}$, 24, $28\frac{1}{3}$, $32\frac{2}{3}$.

Example 3. How many terms of the series $15+13+11 \ldots$ make 55?

Here $s=55,\ a=15,\ d=\text{2nd}-\text{1st}=-2$; n is required.

Now $$s=\frac{n}{2}\{2a+(n-1)d\};$$

$$\therefore\ 55=\frac{n}{2}\{30-2(n-1)\}=\frac{n}{2}\{32-2n\}=16n-n^2.$$

$$n^2-16n+55=0;$$

$$\therefore\ n=5 \text{ or } 11.$$

It will be found on trial that the sum of 11 terms = the sum of 5 terms = 55.

Example 4. The sum of 5 terms in A.P. is 35: find the middle term.

To express 5 terms in A.P. so that their sum shall be as simple as possible, we may write them down as $a-2d,\ a-d,\ a,\ a+d,\ a+2d$.

In this case $$5a=\text{the sum}=35;$$

$$\therefore\ \text{the middle term } a=7.$$

Examples. XLII.

1. **Write down** the 5th term of the series whose nth term is $n+2$.
2. .... 13th .... $3n+1$.
3. .... 12th .... $2n-1$.
4. .... 25th .... $3n-5$.
5. .... 4th .... $7n-4$.
6. .... 1st .... $9n-5$.
7. .... 11th .... $\dfrac{3n-5}{4}$.
8. .... 7th ....$(n+1)$th.... $n+2$.
9. .... 10th .... $3n-1$.
10. .... 14th .... $4n-7$.
11. .... 7th ....$(n+2)$th.... $n+3$.
12. .... 9th .... $3n-1$.
13. .... 1st .... $4n-3$.
14. Find the 12th term of $3+7+11+$ etc.
15. ,, 21st term of $6+4+2+$ etc.
16. ,, 13th term of $\frac{1}{2}+\frac{3}{4}+1+$ etc.
17. ,, 101st term of $20+23+26+$ etc.
18. ,, 17th term of $100+91+82+$ etc.
19. ,, nth term of $1+3+5+$ etc.

Find the last term and the sum of the following series:

20. $1+2\frac{1}{4}+3\frac{1}{2}+$ etc., to 12 terms.
21. $1+1\frac{3}{4}+2\frac{1}{2}+$ etc., to 12 terms.
22. $\cdot32+\cdot29+\cdot26+$ etc., to 50 terms.
23. $\cdot5+\cdot75+1+$ etc., to 20 terms.
24. $\cdot7+\cdot6+\cdot5+$ etc., to 31 terms.

Sum the series

25. $13 + 12\frac{1}{3} + 11\frac{2}{3} +$ etc., to 40 terms.

26. $17 + \frac{49}{3} + \frac{47}{3} +$ etc., to 51 terms.

27. $-7 - 5\frac{2}{3} - 4\frac{1}{3} -$ etc., to 21 terms.

28. $1{\cdot}2 + 2{\cdot}4 + 3{\cdot}6 +$ etc., to 16 terms.

29. $(3a - 2b) + (4a - 5b) + (5a - 8b) +$ etc., to $2n$ terms.

30. $(x+1) + (x+3) + (x+5) +$ etc., to 15 terms.

Find the last term and the sum of

31. $3 + 9 + 15 + \ldots$ to 13 terms.

32. $2 + 7 + 12 + \ldots$ to 11 terms.

33. $21 + 18 + 15 + \ldots$ to 17 terms.

34. $2 - 5 - 12 - \ldots$ to 12 terms.

Find the sum of

35. $1 + 8 + 15 + \ldots$ to 40 terms.

36. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \ldots$ to 9 terms.

37. $\frac{1}{2} + \frac{7}{10} + \frac{9}{10} + \ldots$ to 20 terms.

38. $1 + 3 + 5 + \ldots$ to 80 terms.

39. $\frac{25}{12} + 2 + \frac{23}{12} + \ldots$ to 14 terms.

40. $\frac{1}{2} - \frac{2}{3} - \frac{11}{6} - \ldots$ to 18 terms.

41. The 7th term of an A.P. is 20, and the 13th is 38; find the series.

42. ... 11th .... 36, ... 20th .. 27; ....

43. ... 10th .... 5, ... 17th .. 54; ....

44. ... 12th .... 10, ... 20th .. 8; ....

45. Sum to n terms the series whose nth term is $4 + 5n$.

46. .... $3 + 7n$.

47. The 5th term of an A.P. is 13, and the 9th is 25. What is the 7th term?

48. The 1st term of an A.P. is 3 and the 20th is 136. Find the common difference and the sum of the first 20 terms.

49. Insert 6 arithmetic means between 11 and 53.

50. ...... 8 .... 5 and 11.

51. ...... 20 .... 35 and -28.

52. ...... 5 .... $x + y$ and $x - y$.

53. Find the arithmetic mean between 17 and 33.

54. .... $\frac{1}{2}$ and $\frac{1}{3}$.

55. Find the sum of 30 consecutive odd numbers of which the last is 127.

56. The 7th term of an A.P. is 30 and the 13th is 42. Find the 1st term and the sum of 12 terms.

57. Find the sum of all the even numbers from 2 to 38 inclusive.

58. The sum of the 8th and 4th terms of an A.P. is 24, and the sum of the 15th and 19th is 68. What is the series?

59. The sum of the first seven terms of an A.P. is 140, and the product of the 1st and 7th is 256. Find the terms.

60. A workman is to be paid 1*s.* for his 1st day's work; 1*s.* 1*d.* for the 2nd day, 1*s.* 2*d.* for the 3rd, and so on. Find how much more he earns in the 2nd week than in the 1st, if he works 6 days in the week.

61. A and B go round the world, of which the circuit is 25668 miles. A goes east 1 mile the 1st day, 2 the 2nd, 3 the 3rd and so on. B goes west uniformly at 20 miles a day. When and where do they meet?

62. There are 40 stones in a row, 1 yard apart. How far does a boy travel in bringing them together one by one at the 1st stone (1) if he begins at the first, (2) if he begins at the last?

63. A travels 2 miles the 1st hour, $2\frac{1}{4}$ the 2nd, $2\frac{1}{2}$ the 3rd, and so on. B travels 4 miles an hour. When and where does A overtake B if they start together?

64. How many strokes does a clock make in 12 hours, if it strikes 1 for the half hours?

65. Sum to 10 terms the series whose nth term is $3n+4$.

66. A man receives 5 shillings the 1st week and $3d.$ more each week than the preceding. What does he get in 20 weeks?

67. How may 5 numbers in A.P. be expressed so as to make their sum as simple as possible? How may 4 numbers in A.P. be expressed?

68. Insert 5 arithmetic means between $a-2b$ and $3a+b$.

69. The last term of an A.P. is 10 times the 1st, and the last but one = the sum of the 4th and 5th. Find the number of terms and show that the common difference = 1st term.

70. The sum of 5 terms of an A.P. is 10, the sum of 17 terms is -17; find the series.

71. Find the A.P. in which the first 10 terms together = 100 and the next 10 terms = 300.

72. The 1st and last of $2n+1$ terms of an A.P. are a and c. Write down the sum and the middle term of the series.

73. The mth and nth terms of an A.P. are p, q. Find the 1st term and common difference.

74. The natural numbers 1, 2, 3, ... n^2 are arranged as a magic square, *i.e.* they are so placed in the compartments of a square that all the vertical columns, the horizontal rows and the diagonals have the same sum. Prove that this sum is $\frac{1}{2}n(n^2+1)$. Show that the first 16 numbers can be so arranged with each diagonal in A.P.

CHAPTER XLIII.

GEOMETRICAL PROGRESSION.

270. A series of quantities is said to be in **Geometrical Progression** when each term is equal to the product of the preceding term, and a constant factor.

This constant factor is called the **common ratio**.

Thus 3, 12, 48, 192, ... form a G.P. in whith the common ratio is 4,

1, -3, 9, -27, ... -3,

20, 10, 5, 2·5, ... $\frac{1}{2}$.

271. *To find the nth term of a* G.P.

Let a be the 1st term, r the common ratio.

The 2nd term is ar,

3rd .... ar^2,

4th .... ar^3, and so on.

Thus the **nth term is ar^{n-1}.**

272. *To find the sum of n terms of a* G.P. *whose first term is a and common ratio r.*

Let s be the sum.

$$s = a + ar + ar^2 + \ldots + ar^{n-1},$$

$$sr = \quad ar + ar^2 + \ldots + ar^{n-1} + ar^n;$$

$\therefore$ by subtraction $s(1-r) = a(1-r^n)$;

$$\therefore \quad \mathbf{s} = \frac{\mathbf{a(1-r^n)}}{\mathbf{1-r}} = \frac{\mathbf{a(r^n-1)}}{\mathbf{r-1}} = \frac{\mathbf{rl-a}}{\mathbf{r-1}},$$ where l is the nth term.

273. The meaning of

The sum of an infinite number of terms of a G.P.

If the common ratio, r, is not numerically less than unity, the terms do not decrease as we proceed with the series.

$\therefore$, in this case, the sum of an infinite number of terms is infinitely great.

But if r is a proper fraction, the terms decrease as we proceed, and it is possible to find a *limit* which their sum cannot exceed however many terms we take, and to which it becomes indefinitely near if we take a sufficiently large number of terms.

This *limit* is called the *sum to infinity.*

274. *To find the sum of an infinite number of terms of a* G.P., *whose common ratio is less than unity.*

[Infinity is represented by ∞.]

If $\mathbf{S}_n$ denote the sum of n terms of the series $a, ar, ar^2 \ldots$,

$$\mathbf{S}_n = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Since r is a proper fraction, $\frac{ar^n}{1-r}$ *continually decreases* as n increases.

Hence, the more terms we take, the more nearly does their sum *approximate to* $\frac{a}{1-r}$.

$\therefore$ $\frac{a}{1-r}$ is called the sum of this series *to infinity*, or as it is written, $S_\infty = \frac{a}{1-r}$.

We might express this thus:

The sum of n terms of the series a, ar, ar² ... (r being numerically less than unity) never exceeds $\frac{a}{1-r}$, but continually approaches and becomes indefinitely near to it as n is indefinitely increased.

Example 1. $0\cdot\dot{7} = 0\cdot777\ldots$ to infinity

$$= \frac{7}{10} + \frac{7}{10^2} + \frac{7}{10^3} + \ldots$$

$$\frac{\frac{7}{10}}{1-\frac{1}{10}} \quad \left(\text{Here } a = \tfrac{7}{10} \text{ and } r = \tfrac{1}{10}.\right)$$

$$= \tfrac{7}{9}.$$

Example 2. $0\cdot6\dot{2}\dot{3} = 0\cdot62323\ldots$ to infinity

$$= \frac{6}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \ldots$$

$$= \frac{6}{10} + \frac{\frac{23}{10^3}}{1 - \frac{1}{10^2}} \quad \left(\text{Here } a = \frac{23}{10^3} \text{ and } r = \frac{1}{10^2}.\right)$$

$$= \frac{6}{10} + \frac{23}{990} = \frac{617}{990}.$$

275. *To insert n geometric means between a and b.*

The 1st term is a, the last is b, and there are altogether $n+2$ terms; $\therefore$ $b = ar^{n+1}$.

From this we obtain the value of r; and the required means are $ar, ar^2, ar^3 \ldots ar^n$.

276. *To find a single geometric mean between a and b.*

Denote it by G.

Then a, G, b are in G.P.

$$\therefore \frac{G}{a} = \text{the common ratio} = \frac{b}{G}; \quad \therefore G^2 = ab;$$

$\therefore$ G $=\sqrt{ab}$. This is called **the Geometric Mean** between a and b. It is obviously the same as the mean proportional, just as terms in G.P. are also in continued proportion.

Example 1. Sum the series $3\frac{3}{8}-2\frac{1}{4}+1\frac{1}{2}-\ldots$ to 7 terms.

It should be noticed that $(-1)^7=-1$.

Here $$a=\frac{27}{8},\quad r=\frac{-2\frac{1}{4}}{3\frac{3}{8}}=-\frac{2}{3}.$$

$$s=\frac{a(1-r^n)}{1-r}=\frac{27}{8}\cdot\frac{1-\left(-\frac{2}{3}\right)^7}{1-\left(-\frac{2}{3}\right)}=\frac{27}{8}\cdot\frac{1-\left(-\frac{2^7}{3^7}\right)}{1-\left(-\frac{2}{3}\right)}$$

$$=\frac{27}{8}\cdot\frac{1+\frac{2^7}{3^7}}{1+\frac{2}{3}}=\frac{27}{8}\cdot\frac{3}{5}\cdot\frac{3^7+2^7}{3^7}$$

$$=\frac{3^7+2^7}{40\times 27}=\text{etc.}$$

Example 2. Sum the same series to 6 terms.

$$s=\frac{a(1-r^n)}{1-r}=\frac{27}{8}\cdot\frac{1-\left(-\frac{2}{3}\right)^6}{1-\left(-\frac{2}{3}\right)}=\frac{27}{8}\cdot\frac{1-\frac{2^6}{3^6}}{1+\frac{2}{3}}$$

$$=\frac{27}{8}\cdot\frac{3}{5}\cdot\frac{3^6-2^6}{3^6}=\frac{3^6-2^6}{40\times 9}=\text{etc.}$$

Example 3. Insert four geometric means between 32 and 1.

Here there are 6 terms, $\therefore\ a=32$, and $ar^5=1$.

$$\therefore\ 32r^5=1,\quad r^5=\tfrac{1}{32},$$
$$\text{and}\quad r=\tfrac{1}{2}.$$

$\therefore$ the required means are 16, 8, 4, and 2.

Examples. XLIII.

Write down, or read off, (1) the common ratio,
(2) the 6th term,
(3) the nth term,

in each of the following series:

1. $3+9+27+\ldots$.

2. $1+3+9+\ldots$.

3. $3+\frac{3}{2}+\frac{3}{4}+\ldots$.

4. $3-\frac{3}{2}+\frac{3}{4}-\frac{3}{8}+\ldots$.

5. $1+\frac{1}{2^2}+\frac{1}{2^4}+\ldots$.

6. $1-\frac{1}{2^2}+\frac{1}{2^4}-\ldots$.

7. $9+3+1+\ldots.$ 8. $8+4+2+\ldots.$

9. $x^{n-1}+x^{n-2}+x^{n-3}+\ldots.$ 10. $x^{n+3}+x^{n+2}+x^{n+1}+\ldots.$

11. $\dfrac{1}{x^{n-1}}+\dfrac{1}{x^{n-2}}+\dfrac{1}{x^{n-3}}+\ldots.$ 12. $\dfrac{1}{x^{n-1}}-\dfrac{1}{x^{n-2}}+\dfrac{1}{x^{n-3}}-\ldots.$

13. Sum the series $1+2+2^2+\ldots$ to 10 terms.

14. .... $1+\dfrac{1}{2}+\dfrac{1}{2^2}+\ldots$ to 8 ..

15. .... $3+\frac{3}{4}+\frac{3}{16}+\ldots$ to 5 ..

16. .... $1-\dfrac{1}{2}+\dfrac{1}{2^2}-\dfrac{1}{2^3}+\ldots$ to 9 ..

17. .... to 8 ....

18. .... $a-ax+ax^2-\ldots$ to n ... .

19. .... $x^{n-1}+x^{n-2}+x^{n-3}\ldots$ to n ....

20. ... . $x^n-x^{n-1}+x^{n-2}-\ldots$ to n ....

21. The 1st term of a G.P. is 3 and the common ratio 2, find the 5th term.

22. ... 1st ... . is $\frac{1}{2}$ .... 3, ... 6th ......

23. ... 2nd .... is $\frac{1}{4}$ . ... $\frac{1}{2}$, ... 8th ......

24. ... 5th ... . is 243 .... 3, ... 1st ......

25. Sum the series $\frac{1}{4}+\frac{1}{2}+1+2+\ldots$ to 18 terms.

26. .... $a+ab^2+ab^4+\ldots$ to x .. ..

27. .... $1-c^3+c^6-\ldots$ to 10 ....

28. $\dfrac{1}{\sqrt{2}}-1+\sqrt{2}-\ldots$ to 10 .

Sum to infinity the following geometric series:

29. $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\ldots.$ 30. $9-6+4-\ldots.$

31. $4+3+\frac{9}{4}+\ldots.$ 32. $16+2+\frac{1}{4}+\ldots.$

33. $\left(\dfrac{a}{b}\right)^{\frac{1}{2}}-\left(\dfrac{b}{a}\right)^{\frac{1}{2}}+\left(\dfrac{b}{a}\right)^{\frac{3}{2}}-\ldots$, b being numerically less than a.

34. Sum the series $\dfrac{1}{\sqrt{3}}+1+\sqrt{3}+\ldots$ to 8 terms.

35. .... $5+3+1\frac{4}{5}+\ldots$ to 6 terms.

36. .... $-3-6-12-\ldots$ to 9 terms.

37. In any G.P. the product of 1st and last term = the product of the 2nd and last but one.

38. Sum the series $9-6+4-\ldots$ to 7 terms.

39. Sum the same series to 6 terms.

40. Sum the series $\sqrt{3}-\sqrt{2}+\dfrac{2}{\sqrt{3}}-\ldots$ to 10 terms.

41. Sum n terms of the series whose rth term is $(-a)^r$.

42. The 2nd term is 6 and the 5th is 48; find the sum of 6 terms.

43. The 3rd term is 8 and the 6th is -1; find the sum of 7 terms.

44. Insert 3 geometric means between 5 and 80.

45. ...... 5 .... 54 and $\frac{27}{32}$.

46. ...... 5 .... 21 and $\frac{448}{243}$.

47. ...... 3 .... 96 and 1536.

48. Find the geometric mean between 3 and 147.

49. .... $\frac{a+b}{a-b}$ and $\frac{a-b}{a+b}$.

50. .... -2 and -1458.

51. Sum the series $1-1+1-1+\ldots$ to n terms.

52. Write down the 5th term of the series whose nth term is 2^n.

53. .... 7th .... 2^n-1.

54. .... 1st .... 3^n-2.

55. .... 4th .... $2+(-3)^n$.

56. .... 1st ....

57. .... 3rd ....

58. .. $\overline{n+1}|$th .... 3^n-1.

59. .. $\overline{n-4}|$th .... a^{n-2}.

60. .... 1st .... 2^{n-3}.

61. .... 3rd .... $\overline{n+1}|$th term is 2^n.

62. .. $\overline{n-2}|$th ....

63. .. $\overline{n+3}|$th .... a^n-b.

64. Taking the number of the term for abscissa and the term for ordinate, plot the successive terms of 8, 6, 4, ... and of 8, 4, 2, ...

State what happens with regard to the magnitude of the nth term in these series when n is made infinitely great.

65. Sum to n terms $1+3+7+15+\ldots+(2^r-1)+\ldots$.

Find the value in vulgar fractions of

66. $\cdot\dot{8}$. 67. $\cdot\dot{6}1\dot{2}$. 68. $\cdot1\dot{2}\dot{3}$. 69. $\cdot24\dot{1}\dot{2}$.

70. If P be the product of n terms in G.P., s their sum, s' the sum of their reciprocals, $P^2=\left(\frac{s}{s'}\right)^n$.

71. Prove that the insertion of geometric means between two numbers can be performed by inserting arithmetic means between their logarithms.

72. Insert 3 geometric means between 7 and 112.

73. .... 2 .... 2 and -54.

74. Sum the series $\cdot14+\cdot0028+\cdot000056+\ldots$ by writing down 8 or 9 terms and adding up. Sum it also as an infinite G.P. Compare results.

75. From 3 numbers in G.P. 3 others in G.P. are subtracted. If the remainders be in G.P., prove that all three series have the same common ratio.

76. The sum of the first two terms of a G.P. is 12, and of the first three terms 39; find the series, and determine if the conditions are satisfied by more than one series.

77. Which term of the series $5+5\sqrt{2}+10+\ldots$ is the same as the 200th term of $\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\ldots$?

78. Prove that the first 50 terms of $17+16\frac{5}{7}+16\frac{3}{7}+\ldots$ are together equal to the infinite series $200+120+72+\ldots$.

79. In the series $2+4+8+16+\ldots$ determine the number of digits in the 250th term by using logarithms.

80. If a, b, c be in G.P. and x, y be the arithmetic means between a, b and between b, c respectively, prove that $\dfrac{a}{x}+\dfrac{c}{y}=2$.

81. Sum the series $x^{b-2c}+x^{b-c}+x^{b}+\ldots$ to n terms.

82. Give the nth term of the series $1+5+13+29+\ldots$.

83. Find the sum of all the products formed by taking any two terms of an infinite G.P., and show that if this sum be $\frac{1}{2}$ the sum of the squares of the terms their common ratio is $\frac{1}{3}$.

84. If $1\frac{7}{9}$ and 1 are the 1st and 3rd terms of a G.P., find the sum to infinity.

CHAPTER XLIV.

HARMONICAL PROGRESSION.

277. DEF. *If* $\dfrac{a}{c}=\dfrac{a-b}{b-c}$ *the quantities* a, b, c *are said to be in Harmonical Progression; and a series is said to be a Harmonical Progression when the above relation is satisfied by every three successive terms of it.*

Or, *three quantities are in* H.P. *when the first is to the third as the first minus the second is to the second minus the third.*

278. *If terms are in* H.P. *their reciprocals are in* A.P.

Let a, b, c be in H.P.

Then
$$\frac{a}{c}=\frac{a-b}{b-c};$$
$$\therefore\ ab-ca=ca-bc;$$
$$\therefore\ \frac{ab-ca}{abc}=\frac{ca-bc}{abc};$$

$$\frac{1}{c}-\frac{1}{b}=\frac{1}{b}-\frac{1}{a},$$

i.e. $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in A.P.

279. From the property proved in the last article we can write down any term of an H.P. whose first three terms are given, by inverting the terms, applying the methods of A.P., and inverting the results.

The insertion of means can be performed in this way.

It must be carefully observed that this method does not give us any formula for summing a series in H.P. Summation may be performed, if necessary, by writing down the terms as above and **adding them up.**

280. *To find the Harmonic Mean between x and y.*

Denote it by H.

Then x, H, y are in H.P.;

$$\therefore \frac{x}{y}=\frac{x-H}{H-y} \text{ (by definition)};$$

$$\therefore xH-xy=xy-yH;$$

$$H=\frac{2xy}{x+y}.$$

A, the Arithmetic Mean, has been found to be $\frac{x+y}{2}$.

G, the Geometric ...., ... $\sqrt{xy}$.

H, the Harmonic ...., ... $\frac{2xy}{x+y}$.

A is familiarly known as the *average* of x and y. G, H might be called the Geometric and Harmonic average.

$$AH=\frac{x+y}{2}\cdot\frac{2xy}{x+y}=xy=G^2;$$

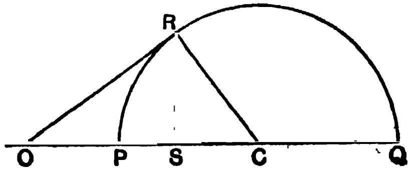
$$\frac{G}{A}=\frac{H}{G},$$

i.e. A, G, H form a G.P. or continued proportion.

Also $A-G=\frac{x+y-2\sqrt{xy}}{2}=\frac{(\sqrt{x}-\sqrt{y})^2}{2}=$ a positive quantity.

A, G, H form a *descending* series.

281. *To represent* A, G, H *graphically.*



In a str. line OPQ mark off OP, OQ to represent the quantities whose mean is required. Bisect PQ at C. On PQ describe a semicircle PRQ. Draw OR a tangent, and RS perp. to PQ.

$$OP + OQ = (OC - CP) + (OC + CQ) = 2 . OC.$$

$\therefore$ OC is the *arithmetic mean* between OP and OQ.

$$OP . OQ = OR^2.$$

$\therefore$ OR is the *geometric mean* between OP and OQ.

$$\frac{2OP . OQ}{OP + OQ} = \frac{2OR^2}{2OC} = \frac{OS . OC}{OC} = OS.$$

OS is the *harmonic mean* between OP and OQ.

Examples. XLIV.

Continue to 6 terms the series

1. $\frac{1}{2}+\frac{4}{9}+\frac{2}{5}+\dots$ **2.** $\frac{1}{8}+\frac{2}{11}+\dots$ **3.** $1+\frac{3}{8}+\frac{3}{13}+\dots$

Find the 12th term of

4. $\frac{3}{2}, \frac{12}{7}, 2 \dots$ **5.** $-\frac{1}{5}, -1, +\frac{1}{3}$. **6.** $\frac{3}{8}, \frac{1}{2}, \frac{3}{4}$.

7. Insert 4 harmonic means between 1 and 6.

8. .. .. 3 .. .. 1 and 20.

9. . .. 4 . ... 2 and 12.

10. ...... 3 .... 9 and $4\frac{1}{5}$.

11. Find the harmonic mean between 2 and $\frac{6}{5}$.

12. If $a-b$, $a-c$, $a-d$ are in H.P., prove that the following three series of quantities are equally so:

(i) $b-c$, $b-d$, $b-a$.

(ii) $c-d$, $c-a$, $c-b$.

(iii) $d-a$, $d-b$, $d-c$.

13. To each of three consecutive terms in G.P. the middle one is added. Prove that the three results are in H.P.

14. If a, b, c are in H.P., then $\frac{1}{a-b}+\frac{1}{b-c}+\frac{4}{c-a}=\frac{1}{c}-\frac{1}{a}$.

15. If $b+c$, $c+a$, $a+b$ are in H.P., prove that a^2, b^2, c^2 are in A.P.

16. Show that the A.M., G.M., and H.M., between x and y are in G.P. If the common ratio of this G.P. be $\frac{3}{5}$, find the ratio of x to y.

17. If b is half the harmonic mean between a and c, prove that

$$a^3 - b^3 + c^3 + 3abc = (a - b + c)^3.$$

18. If the arithmetic mean between two numbers $= 1$, their harmonic mean is the square of their geometric mean.

19. If between two given numbers there be inserted two arithmetic means A_1, A_2, two geometric means G_1, G_2, and two harmonic means H_1, H_2, prove that

$$\frac{G_1 G_2}{H_1 H_2} = \frac{A_1 + A_2}{H_1 + H_2}.$$

20. If a, b, c are in H.P., then $\frac{b+c}{a}$, $\frac{c+a}{b}$, $\frac{a+b}{c}$ are in A.P. Prove this and the converse.

21. Prove that $bc - ad$ is positive, zero, or negative according as a, b, c, d are in A.P., G.P., or H.P.

CHAPTER XLV.

MISCELLANEOUS SERIES, PILES OF SHOT, AND MATHEMATICAL INDUCTION.

282. *To find the sum* (S_1) *of the first n natural numbers.*

These form an A.P.

$$\therefore \; S_1 = 1 + 2 + 3 + \ldots + n = \frac{n}{2}(n + 1) \quad \left[S = \frac{n}{2}(a + l)\right].$$

283. *To find the sum* (S_2) *of the squares of the first n natural numbers.*

$$n^3 - (n - 1)^3 = 3n^2 - 3n + 1,$$
$$(n - 1)^3 - (n - 2)^3 = 3(n - 1)^2 - 3(n - 1) + 1,$$
$$(n - 2)^3 - (n - 3)^3 = 3(n - 2)^2 - 3(n - 2) + 1$$

$$2^3 - 1^3 = 3 \, . \, 2^2 - 3 \, . \, 2 + 1,$$
$$1^3 - 0^3 = 3 \, . \, 1^2 - 3 \, . \, 1 + 1\,;$$

$$\therefore \text{ by addition } n^3 = 3S_2 - 3S_1 + n\,;$$

$$\therefore \; 3S_2 = n^3 - n + 3S_1 = n(n^2 - 1) + \frac{3n(n+1)}{2}$$

$$= \frac{n \, . \, (n+1)}{2}\{2(n - 1) + 3\} = \frac{n(n+1)(2n+1)}{2}\,;$$

$$\therefore \; S_2 = \frac{n(n+1)(2n+1)}{6}.$$

284. *To find the sum* (S_3) *of the cubes of the first n natural numbers.*

$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1,$$

$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$$

....

$$2^4 + 1^4 = 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1,$$

$$1^4 - 0^4 = 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1;$$

$$\therefore \text{ by addition } n^4 = 4S_3 - 6S_2 + 4S_1 - n;$$

$$\begin{aligned} \therefore \quad 4S_3 &= n^4 + n + 6S_2 - 4S_1 \\ &= n(n^3+1) + n(n+1)(2n+1) - 2n(n+1) \\ &= n(n+1)\{n^2 - n + 1 + 2n + 1 - 2\} \\ &= n^2(n+1)^2; \end{aligned}$$

$$\therefore \quad S_3 = \left[\frac{n(n+1)}{2}\right]^2.$$

285. *To find the sum of the series*

$$a + (a+b)x + (a+2b)x^2 + (a+3b)x^3 + \dots \textit{ to n terms.}$$

Let S denote the sum.

$$S = a + (a+b)x + (a+2b)x^2 + (a+3b)x^3 + \dots + (a + \overline{n-1}\,.b)x^{n-1};$$

$$\therefore \; Sx = \qquad ax + (a+b)x^2 + (a+2b)x^3 + \dots + (a+\overline{n-2}\,.b)x^{n-1} + (a+\overline{n-1}\,.b)x^n.$$

$\therefore$ by subtraction

$$\begin{aligned} S(1-x) &= a + bx + bx^2 + bx^3 + \dots + bx^{n-1} - (a + \overline{n-1}\,.b)x^n \\ &= a + \frac{bx(1-x^{n-1})}{1-x} - ax^n - (n-1)bx^n \\ &= a(1-x^n) - (n-1)bx^n + \frac{bx(1-x^{n-1})}{1-x}; \end{aligned}$$

$$\therefore \quad S = \frac{a(1-x^n) - (n-1)bx^n}{1-x} + \frac{bx(1-x^{n-1})}{(1-x)^2}.$$

If x is a proper fraction the sum to infinity is

$$\frac{a}{1-x} + \frac{bx}{(1-x)^2},$$

for x^n diminishes continually as n increases.

Example 1. Find $\sum\limits_{r=1}^{r=n} (3r^2+5r)$,

i.e. sum to n terms the series whose rth term is $3r^2+5r$.

The 1st term $=3.1^2+5.1$,

... 2nd term $=3.2^2+5.2$,

etc. ....

$$\therefore \text{ the sum} = 3(1^2+2^2+3^2+\ldots+n^2)+5(1+2+3+\ldots+n)$$

$$=3.\frac{n(n+1)(2n+1)}{6}+5.\frac{n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}(2n+1+5)=n(n+1)(n+3).$$

Example 2. Sum to n terms $1.4.5+2.5.6+3.6.7+\ldots$.

The nth term $=n(n+3)(n+4)=n^3+7n^2+12n$.

$\therefore$ the sum

$$=1^3+2^3+3^3+\ldots+n^3+7(1^2+2^2+3^2+\ldots+n^2)+12(1+2+3+\ldots+n)$$

$$=\frac{n^2(n+1)^2}{4}+\frac{7n(n+1)(2n+1)}{6}+\frac{12n(n+1)}{2}$$

$$=\frac{n(n+1)}{2}\left\{\frac{n^2+n}{2}+\frac{14n+7}{3}+12\right\}$$

$$=\frac{n(n+1)}{2}\cdot\frac{3n^2+3n+28n+14+72}{6}$$

$$=\frac{n(n+1)(3n^2+31n+86)}{12}.$$

PILES OF SHOT.

286. *To find the number of shot in a complete pyramidal pile of n layers on a triangular base.*

The top layer consists of 1 ball,

the 2nd .... $1+2$ balls,

the 3rd .... $1+2+3$....

The rth layer from the top consists of $1+2+3+\ldots r$, *i.e.* $\frac{r(r+1)}{2}$.

$\therefore$ we have to find $\Sigma\{\frac{1}{2}(r^2+r)\}$ from $r=1$ to $r=n$.

This expression $=\frac{1}{2}(1^2+2^2+3^2+\ldots+n^2)+\frac{1}{2}(1+2+3+\ldots+n)$

$$=\frac{1}{2}\cdot\frac{n.\overline{n+1}.\overline{2n+1}}{6}+\frac{1}{2}\cdot\frac{n.\overline{n+1}}{2}$$

$$=\frac{n.\overline{n+1}}{12}\{2n+1+3\}=\frac{n.\overline{n+1}.\overline{n+2}}{6}.$$

Example. An incomplete triangular pile has 12 shot in a side of the base, and 5 in a side of the top. Find the number of shot.

The number is the difference between the complete pile and the missing upper part, the missing part being a complete triangular pile with 4 shot in a side of its base.

$$\therefore \text{ the reqd. number} = \frac{12 \cdot 13 \cdot 14}{6} - \frac{4 \cdot 5 \cdot 6}{6}$$
$$= 364 - 20 = 344.$$

287. *To find the number of shot in a complete pyramidal pile n layers of shot on a square base.*

The top layer consists of 1,

the 2nd .... 2^2,

the 3rd .... 3^2,

the rth .... r^2.

$$\therefore \text{ the total number} = 1^2 + 2^2 + 3^2 + \ldots + n^2$$
$$= \frac{n \cdot \overline{n+1} \cdot \overline{2n+1}}{6}.$$

288. *To find the number of shot in a completed pile arranged on a rectangular base.*

Let the number of shot in the base be m along the length and n along the breadth.

The number in the base is mn.

The number in the layer next above is $(m-1)(n-1)$, and at the top the pile finishes with a row of $m - n + 1$ shot.

The number of layers is n.

The total number

$$= 1(m-n+1) + 2(m-n+2) + 3(m-n+3) + \ldots \text{ to } n \text{ terms}$$
$$= (m-n)(1+2+3+\ldots+n) + 1^2 + 2^2 + 3^2 + \ldots + n^2$$
$$= (m-n)\frac{n(n+1)}{2} + \frac{n(n+1)(2n+1)}{6}$$
$$= \frac{n}{6}(n+1)\{3m - 3n + 2n + 1\} = \frac{n}{6}(n+1)(3m - n + 1).$$

289. The following should be studied with care. Such rearrangements enable us to sum many series whose terms are fractions of which the denominators follow certain laws.

$$\frac{1}{n \cdot \overline{n+1}} = \frac{1}{n} - \frac{1}{n+1}. \qquad \frac{1}{x(x+a)} = \frac{1}{a}\left(\frac{1}{x} - \frac{1}{x+a}\right).$$

$$\frac{1}{n \cdot \overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3}} = \frac{1}{(n+1)(n+2)}\left[\frac{1}{n(n+3)}\right]$$

$$= \frac{1}{3(n+1)(n+2)}\left[\frac{1}{n} - \frac{1}{n+3}\right].$$

$$\frac{1}{x(x+3a)(x+5a)(x+7a)} = \frac{1}{(x+3a)(x+5a)}\left[\frac{1}{x(x+7a)}\right]$$

$$= \frac{1}{7a(x+3a)(x+5a)}\left[\frac{1}{x} - \frac{1}{x+7a}\right].$$

In each case one term is expressed as the *difference of two expressions*

Example. Sum the series $\frac{1}{1 \cdot 4} + \frac{1}{4 \cdot 7} + \frac{1}{7 \cdot 10}$ ... to n terms and to infinity.

$$\text{The first term} = \frac{1}{3}\left(1 - \frac{1}{4}\right),$$

$$\text{...... second ...} = \frac{1}{3}\left(\frac{1}{4} - \frac{1}{7}\right),$$

$$\text{... .. third ...} = \frac{1}{3}\left(\frac{1}{7} - \frac{1}{10}\right),$$

... .

$$\text{...... } n\text{th ....} = \frac{1}{3}\left(\frac{1}{3n-2} - \frac{1}{3n+1}\right).$$

$\therefore$ by addition, the sum of n terms $= \frac{1}{3}\left(1 - \frac{1}{3n+1}\right)$.

Also, as n continually increases, $\frac{1}{3n+1}$ continually diminishes, and ultimately may be taken as zero.

$\therefore$ the sum to infinity $= \frac{1}{3}$.

Examples. XLV. a.

1. Sum to n terms the series whose nth term is $n^2 - 1$.
2. Sum the series $1 + 2x + 3x^2 + 4x^3$... to n terms. Test the result.
3. Which term of the series 7, 11, 15, ... is 107?
4. Sum the squares of the odd terms of the series
$$a + ar + ar^2 + \ldots + ar^{2n-1}.$$
5. Sum to n terms the series whose nth term is $n^3 - n$.
6. .... $3n^2 - n$.
7. .. .. $n^2 + 3n + 2$.

8. Sum to n terms the series whose nth term is an^2+bn.

9. ... . $\frac{n^2}{2}-\frac{1}{6}$.

10. Find the series whose sum to n terms is an^2.

11. Write down the nth terms of the following series:

$$1^2+2^2+3^2+\dots$$

$$\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\dots,$$

and shew that they form neither arithmetic nor geometric progressions.

12. Sum $\frac{5}{6}+\frac{5}{9}+\frac{10}{27}+\dots$ to n terms and to infinity.

13. ...... $6-5+\frac{25}{6}-\dots$ to $2n$ terms and to infinity.

14. ...... $1-3+5-7+\dots$ to $2n$ terms.

15. .. . $x^2+(x+1)^2+(x+2)^2+$ to n terms.

16. .. $12^2+13^2+14^2+\dots$ to 10 terms.

17. If S_n is the sum of n terms of $a+d$, $a+2d$, ..., find the value of $S_{n+4}-2S_{n+3}+S_{n+2}$.

18. The sum of the squares of the first n odd integers $=\frac{n}{3}(2n-1)(2n+1)$.

19. Sum to n terms the series whose nth term is 2^{2n} $2n$.

Write down, or read off the nth term of each of the following series:

20. $1.2+2.3+3.4+\dots$

21. $1.3+3.5+5.7+\dots$

22. $1.2.3+2.3.4+3.4.5+\dots$

23. $a+(a+2)x+(a+4)x^2+\dots$

24. $1.3.5+3.5.7+5.7.9+\dots$

25. $1+5+10+17+26+\dots$

26. $1.2-2.3+3.4-4.5+\dots$

27. $5.7+7.9+9.11+\dots$

28. Find the nth term and the sum of n terms of the series

$$1+3+6+10+15+21+\dots$$

[Notice that the 2nd term is $1+2$, 3rd term $1+2+3$, and so on.]

Sum to n terms the series

29. $1.2+2.3+3.4+\dots$

30. $1.3+3.5+5.7+\dots$

31. $1.2.3+2.3.4+3.4.5+\dots$

32. $a+(a+1)x+(a+2)x^2+\dots$

33. $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\dots$

34. $1.3.5+3.5.7+5.7.9+\dots$

35. Sum $\frac{1}{2}-\frac{2}{2^2}+\frac{3}{2^3}-\frac{4}{2^4}+\dots$ to infinity.

Write down the nth term of the following series and so sum them to n terms:

36. $1^2.3+2^2.5+3^2.7+\dots$

37. $1.4+4.7+7.10+\dots$

38. Sum the series $1+3x+5x^2+7x^3+\dots$

(a) to n terms, (b) to infinity, x being less than unity.

39. Find the sum of n terms of the series

$$1+\frac{2}{2}+\frac{3}{2^2}+\frac{4}{2^3}+\dots$$

Sum the following series to infinity:

40. $1+\frac{4}{2}+\frac{7}{2^2}+\frac{10}{2^3}+\dots$. **41.** $1-\frac{4}{2}+\frac{7}{2^2}-\frac{10}{2^3}+\dots$.

42. Prove that the sum of n terms of the series $1+\frac{7}{5}+\frac{13}{5^2}+\frac{19}{5^3}+\dots$ is always less than $3\frac{1}{8}$ however great n may be.

Sum the following series:

43. $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\dots$ (*a*) to n terms, (*b*) to infinity.

44. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\dots$ ....

45. $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\dots$ ....

46. $\frac{1}{1.3}+\frac{1}{2.4}+\frac{1}{3.5}+\dots$ ....

47. $\frac{1}{x(x+a)}+\frac{1}{(x+a)(x+2a)}+\frac{1}{(x+2a)(x+3a)}+\dots$ to n terms.

48. $1.2^2+2.3^2+3.4^2+4.5^2+\dots$ to n terms.

49. $\frac{4}{1.2.3}+\frac{5}{2.3.4}+\frac{6}{3.4.5}+\frac{7}{4.5.6}+\dots$ to n terms.

(Write down the nth term and show that it $=\frac{1}{n.\overline{n+1}}+\frac{1}{n.\overline{n+1}.\overline{n+2}}$.)

PILES OF SHOT.

50. Find the number of shot in a completed triangular pile of 8 courses.

51. .... 10 ....

52. .... 15 ....

53. Find the number of shot in a completed square pile of 9 courses.

54. .... 12 ....

55. If the base is a rectangle of length 10 and breadth 7 shot, what is the number of shot in the top row if the pile is complete?

56. What is the number in the pile?

57. If a triangular pile has 6 courses, the base having 9 shot in a side, what is the number in the incomplete pile?

58. If there are 64 shot in the base of a square pile, what is the number in the 4 lowest courses?

59. In a triangular pile how many shot are there in the 7th layer counting from the top? How many shot are there above this layer?

60. The top of a rectangular pile is a line of 6 shot, and the bottom layer contains 66; how many layers are there; and how many shot in all?

61. The lowest course of a square pile contains 49 shot, and there are 3 layers; how many shot are required to complete it?

62. Find the number in an incomplete square pile, the top layer containing 529, and the bottom 5184.

MATHEMATICAL INDUCTION.

290. The following examples will explain the method known by this name.

Example 1. To prove that the sum of the first n natural numbers

$$=\frac{n(n+1)}{2}.$$

Let S_n denote the sum of n terms, and *assume* that

$$S_n=\frac{n(n+1)}{2}.$$

Then $S_{n+1}=S_n+n+1=\frac{n(n+1)}{2}+n+1=(n+1)\left(\frac{n}{2}+1\right)$

$$=\frac{(n+1)(n+2)}{2};$$

$\therefore$ if the formula holds for n terms, it also holds for $n+1$ terms.

But $$S_2=1+2=\frac{2 \cdot 3}{2};$$

$\therefore$ the formula is true for 2 terms;

$\therefore$ it is also true for 3 terms;

$\therefore$ .... 4 ...., and so on;

$$\therefore\ 1+2+3+\ldots+n=\frac{n(n+1)}{2}.$$

Example 2. To prove that

$$\frac{1}{1 \cdot 3}+\frac{1}{3 \cdot 5}+\frac{1}{5 \cdot 7}\ldots \text{ to } n \text{ terms}=\frac{n}{2n+1}.$$

Let S_n denote the sum of n terms, and *assume* that

$$S_n=\frac{n}{2n+1}.$$

The nth term $$=\frac{1}{(2n-1)(2n+1)},$$

$$\therefore \text{ the } n+1\text{th term}=\frac{1}{(2n+1)(2n+3)};$$

$$\therefore\ S_{n+1}=\frac{n}{2n+1}+\frac{1}{(2n+1)(2n+3)}=\frac{2n^2+3n+1}{(2n+1)(2n+3)}$$

$$=\frac{(2n+1)(n+1)}{(2n+1)(2n+3)}=\frac{n+1}{2n+3};$$

$\therefore$ if the formula is true for n terms, it is also true for $n+1$ terms.

But $$S_2=\frac{1}{1 \cdot 3}+\frac{1}{3 \cdot 5}=\frac{5+1}{3 \cdot 5}=\frac{2}{5}=\frac{2}{2 \cdot 2+1}.$$

i.e. the formula is true for 2 terms;

$\therefore$ .... 3 ....

$\therefore$ .... 4 .. .. and so on;

$$\therefore\ S_n=\frac{n}{2n+1}.$$

Examples. XLV. b.

Prove the following by Mathematical Induction:

1. The sum of n terms of an A.P. $=\frac{n}{2}\{2a+(n-1)d\}$.

2. $1+3+5+\ldots$ to n terms $=n^2$.

3. a^n-b^n is divisible by $a-b$.

4. $\frac{1}{1\,.\,2}+\frac{1}{2\,.\,3}+\frac{1}{3\,.\,4}+\ldots$ to n terms $=1-\frac{1}{n+1}$.

5. $1^2+2^2+3^2+\ldots$ to n terms $=\frac{n(n+1)(2n+1)}{6}$.

6. $1^3+2^3+3^3+\ldots$ to n terms $=\frac{n^2(n+1)^2}{4}$.

7. $1\,.\,2+2\,.\,3+3\,.\,4+\ldots$ to n terms $=\frac{n(n+1)(n+2)}{3}$.

8. The sum of n terms of a G.P. $=\frac{a(1-r^n)}{1-r}$.

9. If n be even, n^2+2n is divisible by 8.

10. If $\phi(n)=n^3+20n$, find $\phi(n+2)-\phi(n)$, and hence prove that n^3+20n is divisible by 48 if n be even.

11. Prove that $1\,.\,3+2\,.\,4+3\,.\,5+\ldots$ to n terms $=\frac{1}{6}n(n+1)(2n+7)$.

12. The amount of £P at compound interest in n years at x per cent. per annum $=\text{PR}^n$, where $\text{R}=1+\frac{x}{100}$.

13. The difference between any integer and its cube is a multiple of 6.

14. $3^{4n}-1$ is divisible by 80.

CHAPTER XLVI.

REVISION PAPERS.

XLVI. a.

1. Simplify $\dfrac{4\sqrt{2}-2\sqrt{3}}{2\sqrt{18}-\sqrt{27}}$.

2. Solve $\sqrt{x+a}+\sqrt{x-a}=\sqrt{2a}$.

3. The sum of three numbers in A.P. is 33 and their product 792. Find them.

4. In a G.P. the sum of the first 3 terms : the sum to infinity $=7:8$. Find the common ratio.

5. If $\dfrac{x}{a+2b+c}=\dfrac{y}{2a+b-c}=\dfrac{z}{4a-4b+c}$, then

$$\frac{a}{x+2y+z}=\frac{b}{2x+y-z}=\frac{c}{4x-4y+z}.$$

6. Find the sum of n terms of a series whose r^{th} term is $ar+b$.

7. If the arithmetic mean between x and y = twice their geometric mean, then $\dfrac{x}{y}=\dfrac{2+\sqrt{3}}{2-\sqrt{3}}$ or the reciprocal of this.

XLVI. b.

1. Find $(\sqrt{3}+2\sqrt{2})^{-1}$ to 4 places of decimals.

2. Solve $\sqrt{x^2-6x+17}+2x^2=12x-13$.

3. If $\dfrac{a+b}{ax+by}=\dfrac{b+c}{bx+cy}=\dfrac{c+a}{cx+ay}$, each ratio $=\dfrac{2}{x+y}$, if $a+b+c$ is not zero.

4. The sum of 3 numbers in H.P. is 11, and the sum of their squares is 49. Find the numbers.

5. In an A.P. the 1st term is 7, the common difference 2, and the sum of n terms 247; find n.

6. Find the product $a \cdot a^2 \cdot a^3 \ldots a^n$.

7. Find by logarithms the value of $4^{\frac{1}{3}} \div 2^{\frac{1}{5}}$, and simplify

$$\log \cdot 06+\log (\cdot 6)^2-\log 4-\log 54.$$

XLVI. c.

1. If z is a mean proportional between x any y,

$$\frac{x}{y}=\frac{x^2+xz+z^2}{y^2+yz+z^2}.$$

2. If the illumination from a light varies inversely as the square of the distance from the light, show that for all points in a straight line between 2 equal lights the total illumination is least midway between them.

3. $\dfrac{x}{b+c-a}=\dfrac{y}{c+a-b}=\dfrac{z}{a+b-c}$; prove that

$$(b-c)x+(c-a)y+(a-b)z=0.$$

4. Find a series in which the sum of n terms shall be $3n^2$ for all values of n.

5. Find the product $a^{\frac{1}{2}}a^{\frac{1}{4}}a^{\frac{1}{8}}$.. to infinity.

6. Sum to n terms the series whose r^{th} is 2^r-r.

7. How many digits are there in the 101st term of $1+5+25+125$?

XLVI. d.

1. Find 4 numbers in A.P. such that the product of the extremes shall be 45, and the product of the means 77.

2. Find the difference of the roots of the equation

$$x^2-2bx+c^2=0.$$

Write down an equation whose roots are each equal to the sum of the reciprocals of the roots of this equation.

3. A cask was filled with wine and water mixed in the ratio 5 : 3. When 8 gallons had been drawn off and the cask filled up with water the ratio was 3 : 5. How many gallons did the cask hold?

4. The first and last of 46 terms in A.P. are -5 and $+25$: find the two middle terms and the sum of all the terms.

5. Find by logarithms the smallest number of terms of the G.P. 4, 6, 9 which have a sum exceeding 8000.

6. If an elastic ball, let fall on a table, always rebounds to a height which is the height from which it falls multiplied by e^2 (e being less than 1), find the total space described by the ball if it is let fall originally from a height h and left to itself.

7. If a, b, c are in H.P., $a(b-c)x^2+b(c-a)xy+c(a-b)y^2$ is a perfect square.

XLVI. e.

1. If $\alpha+\beta=-p$, and $\alpha\beta=q$ find in terms of p and q the equation whose roots are $\alpha+2\beta$ and $\beta+2\alpha$.

2. At present B's age : A's age $=5:2$, but in 30 years the ratio will be 35 : 23. Find their ages.

3. Solve to 2 places of decimals

$$5^x \cdot 7^{x-1}=11^{x+2}.$$

4. Sum to n terms $2^2+4^2+6^2+\ldots$

5. Find an infinite G.P. in which each term is 3 times the sum of all that follow it.

6. Three regular polygons of p, q, r sides respectively fit round a point exactly: prove that

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}=\frac{1}{2}.$$

7. Prove that $\dfrac{a+2c}{b+2d}$ is intermediate in value between $\dfrac{a}{b}$ and $\dfrac{c}{d}$. Illustrate this also by a figure.

XLVI. f.

1. Nine numbers $\begin{matrix} a & b & c \\ d & e & f \\ g & h & k \end{matrix}$ are arranged in a square so that each horizontal line, each vertical line, and each diagonal line has a sum equal to x. Prove that $e=\frac{x}{3}$, and $a+c+k+g=\frac{4x}{3}$.

2. On a cylinder, whose length is 6 inches and circumference 3 inches, is a screw whose thread makes 4 turns in the length; on another cylinder exactly equal a thread makes 3 turns. Find the ratio of the lengths of the threads.

3. Sum to n terms $1.3+3.5+5.7+\ldots$.

4. Test which is the greater $\sqrt{17}$ or $2\sqrt[4]{18}$.

5. Find by logarithms the value of $(66901\times 337\div 7824)^{\frac{1}{5}}$.

6. Prove that $\log 20+7\log\frac{15}{16}+5\log\frac{24}{25}+3\log\frac{80}{81}=1$.

7. The circumference of a circle varies as the diameter, and when the diameter is $\frac{1}{2}$ foot the circumference is 1·5708 feet. Find to 3 places the length of an arc subtending 3° at the centre of a circle of 50 feet diameter.

XLVI. g.

1. The sum of all the products in a multiplication table extending as far as $n\times n=\left(\frac{n.\overline{n+1}}{2}\right)^2$.

2. The angles of a triangle are in A.P. What is the middle one?
.... pentagon ... .

3. The natural numbers are grouped thus; 1; 2, 3; 4, 5, 6; etc. .... Prove that the sum of the n^{th} group $=\frac{n(n^2+1)}{2}$.

4. Find by logarithms an approximate value for the irrational root of the equation $3^{2x}-5\times 3^x+6=0$.

5. If x is real $\frac{x^2-4x-20}{x-7}$ cannot have a real value between 8 and 12.

6. Three spheres, of radii 3, 4, 5 cm., are melted and formed into a sphere. What is its radius, if the volume of a sphere $\propto$ the cube of the radius?

7. Draw the graph of $xy=4$, and find from the figure where it is met by the line $x+y=4$. How are these lines related?

XLVI. h.

1. Between what powers of 10 do the following lie: 37^2, 37^3, 37^5, $37^{\frac{1}{10}}$, $37^{\frac{1}{40}}$.

2. Arrange in order of magnitude $5^{\frac{1}{5}}$, $6^{\frac{1}{6}}$, $7^{\frac{1}{7}}$; and find their product to 2 places of decimals.

3. Find the least positive value of $4x+\frac{9}{x}$.

4. The sum of 5 terms of an A.P. is 10, and the sum of 17 terms is -17; find the series.

5. The plan of an estate is drawn to a scale of 10 inches to a mile: Find the area of the part which represents 20 acres.

6. How many integers between 200 and 300 are divisible by 7? Find their sum.

7. If the time of travelling any distance $\propto \sqrt{\text{distance}}$, and the time for 144 feet be 3 seconds, find the time for 400 feet.

XLVI. i.

1. Find from the tables the logarithms of 400, 401, 402, 403, 404, 405, 406. Find the excess of each over log 400. On squared paper represent the increments of the logarithms corresponding to the addition of 1, 2, 3, 4, 5, 6 to the number 400. Show how to find, with the help of your diagram, log 403·2, log 403·5, log 403·3.

2. Sum to n terms and to infinity the series $\frac{1}{1.4}+\frac{1}{2.5}+\frac{1}{3.6}+\ldots$.

3. Two companies appeared in the field in strength as 9 to 11. At the end of the day the relative strength was 5 to 8. Of the men in the companies 35 per cent. were missing, and 30 men of these belonged to the 2nd company. Find the number in each company at first.

4. Insert 5 arithmetic means between $a-2b$ and $3a+b$.

5. The sum of 7 terms of an A.P. is 147, and the product of the 1st and last is 297. Find the common difference.

6. The geometric mean of two numbers is $\frac{1}{2}$, and the harmonic mean $\frac{2}{5}$. Find the numbers.

7. If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^2-b^2+c^2-d^2}{(a-b)^2+(c-d)^2}=\frac{(a+c)^2-(b+d)^2}{(a-b+c-d)^2}$.

XLVI. k.

1. What number must be subtracted from each of the numbers 8, 10, 13, 17, that the remainders may be in proportion?

2. Find the number of digits in the product of 2^{19} by 3^{17}, and find the first 2 figures of the product.

3. In a series of right-angled triangles, one side has, in succession, the lengths 1×4 in., 2×6 in., 3×8 in., 4×10 in., etc., and the hypotenuse differs in length from this side by 1 inch: show that the lengths of the other side form an A.P.

4. Find the square root of $3x-1+2\sqrt{2x^2+x-6}$.

5. If the m^{th} term of a harmonic series be n, and the n^{th} term m, prove that the $(m+n)^{\text{th}}$ term is $\frac{mn}{m+n}$.

6. Show that the arithmetic means between a and b, between $\sqrt{\frac{a}{b}}$ and $\sqrt{\frac{b}{a}}$, and between $\frac{1}{a}$ and $\frac{1}{b}$ are in G.P.

7. Simplify the n^{th} term, and find the sum of n terms of the series $1/1+1/(1+2)+1/(1+2+3)+\ldots$.

XLVI. l.

1. The sides of a right-angled triangle are in A.P. Show that they are proportional to 3, 4, 5.

2. If $x \propto y+z$, and $y \propto z+x$, then $x \propto y$.

3. Solve $15^{2x} . 5^{x-4} . 11^{x-2} = 7^{x-1}$.

4. If n is odd, the sum of $1+2^2+3+4^2+5+6^2+\ldots$ to n terms

$$=\tfrac{1}{12}(n+1)(2n^2+n+3).$$

5. If P, Q, R are the p^{th}, q^{th}, r^{th} terms of an A.P., prove that

$$P(q-r)+Q(r-p)+R(p-q)=0.$$

6. How many balls are there in an incomplete triangular pile of 6 courses, the top course containing 210 balls.

7. The area of a $\triangle$ is $\sqrt{s(s-a)(s-b)(s-c)}$ where s = semiperimeter. Find by logarithms the area of a $\triangle$ whose sides are 606, 121, 725 yards.

XLVI. m.

1. The incomes of A and B are as 5 to 3, and their expenditures as 7 to 4. Each saves £300 a year. Find their incomes.

2. Which term of the series 1, 3, 5, 7 ... is a mean proportional between the 23rd and 63rd terms?

How many terms of the same series have their sum equal to one fourth of the sum of the first 20 terms?

3. The perimeter of a right-angled $\triangle$ is 15 feet, and the hypotenuse exceeds one of the sides by $2\frac{1}{2}$ feet. Find the 3 sides.

4. Find the sum of 9 terms of the series

$$2a+a\sqrt{2}+a+\frac{a}{\sqrt{2}}+\ldots.$$

What value must a have so that the sum *ad inf.* may be 8?

5. Prove $\log_a b \times \log_b a = 1$. Find $\log_4 1000$ from $\log_{10} 2$.

6. 105 halfpenny pieces lying on a flat surface with their edges in contact are just contained by a frame in the form of an equilateral $\triangle$. The diameter of a halfpenny being 1 inch, show that the side of the $\triangle$ is $(13+\sqrt{3})$ inches.

7. A man starts on a journey of 9 miles at 3 miles an hour, and increases his pace by $\frac{1}{4}$ mile an hour each quarter of an hour. How long does he take?

CHAPTER XLVII.

SCALES OF NOTATION.

291. In the ordinary notation of Arithmetic the value of a digit depends on its position with regard to the other digits of the same number.

Thus in the number 5837 there are 7 units, 3 tens, 8 hundreds and 5 thousands. In the number ·614 there are 6 tenths, 1 hundredth and 4 thousandths.

The number 5837·614 might be written in the form

$$5 \times 10^3 + 8 \times 10^2 + 3 \times 10 + 7 + \frac{6}{10} + \frac{1}{10^2} + \frac{4}{10^3}.$$

This is the common scale of notation, the **decimal or denary scale**, and 10 is called the **radix** of the scale. Any other number might be employed instead of 10 for the radix of a scale.

If the scale of notation were r, then 5837·614 would mean

$$5r^3 + 8r^2 + 3r + 7 + \frac{6}{r} + \frac{1}{r^2} + \frac{4}{r^3}.$$

In the scale of 10 the number ·614 is called a decimal fraction.

In any other scale it must be called a **radix fraction.**

In any scale the largest digit which may occur is less by 1 than the radix of the scale. In the scale of 12, known as the duodenary scale, ten and eleven may occur as well as the ordinary digits; and it is necessary to denote them by single symbols. Thus t denotes 10, e 11.

If 263 denotes a number in the scale of 8, the number is equivalent to $2 \times 8^2 + 6 \times 8 + 3$, *i.e.* 179 in the denary scale.

292. *To convert a number into the scale of r.*

Let N be the number.

Let e, d, c, b, a be the digits, in the order in which they occur from left to right, in the new scale.

Then $N = er^4 + dr^3 + cr^2 + br + a$.

Divide through by r.

Let N_1 be the integral quotient when N is divided by r.

Then $N_1 = er^3 + dr^2 + cr + b$, remainder a.

Let N_2 be the integral quotient when N_1 is divided by r, and so on.

$N_2 = er^2 + dr + c$, remainder b.

$N_3 = er + d$, remainder c.

$N_4 = e$, remainder d.

Thus the rule is found: **Divide the given number repeatedly by the radix of the scale into which it is to be converted, and**

take the successive remainders in order for the digits, the 1st remainder being the digit in the units place.

The number as originally expressed may be in any scale. It is only necessary to perform the division with due regard to the scale in which it is. An example will make this clear.

(1) Convert 5647 from the scale of 10 to the scale of 8.

(2) Convert 5647 from the scale of 9 to the scale of 8.

The working is as follows:

(1)	(2)
8 | 5647	8 | 5647
8 | 705...7	8 | 638...6
8 | 88...1	8 | 72...1
8 | 11...0	8 | 8...1
1...3	1...0
Result 13017.	Result 10116.

In (2) the division is performed thus:

56 in scale $9 = 5 \times 9 + 6 = 51$ in the common scale.

8 into 51 goes 6 times and 3 over.

34 in scale $9 = 3 \times 9 + 4 = 31$ in the common scale.

8 into 31 goes 3 times and 7 over.

77 in scale $9 = 7 \times 9 + 7 = 70$.

8 into 70 goes 8 times and 6 over.

Thus we obtain 638...6.

Each division is worked in the same way.

We may test each of these results by converting the numbers back into their original scales.

(1)	(2)
10 | 13017	9 | 10116
10 | 1064...7	9 | 717...7
10 | 70...4	9 | 63...4
5...6	5...6
5647.	5647.

Example. Multiply 3651 by 523 in the scale of 7 (septenary scale).

```
   3651
    523
-------
  25545
 10632
14613
-------
3012033
```

It will be sufficient to explain the 1st line of the multiplication.

3651×5.

$1 \times 5 = 5$ which is put down.

$5 \times 5 = 25 = 3 \times 7 + 4$. Put down 4 and carry 3.

$6 \times 5 + 3 = 33 = 4 \times 7 + 5$. Put down 5 and carry 4.

$3 \times 5 + 4 = 19 = 2 \times 7 + 5$. Put down 25.

In the 3rd column of the addition $6 + 3 + 5 = 14 = 2 \times 7 + 0$.

In the 4th column 2 (carried) $+ 4 + 6 + 4 = 16 = 2 \times 7 + 2$.

In the 5th column 2 (carried) $+ 1 + 0 + 5 = 8 = 1 \times 7 + 1$.

In the 6th column 1 (carried) $+ 1 + 5 = 7 = 1 \times 7 + 0$.

In the 7th column 1 (carried) $+ 2 = 3$.

293. *To convert a decimal into a radix fraction.*

Let D represent the decimal, and *abcd* the radix fraction which is in the scale of r.

$$D = \frac{a}{r} + \frac{b}{r^2} + \frac{c}{r^3} + \frac{d}{r^4}.$$

By multiplying by r we get $a + \frac{b}{r} + \frac{c}{r^2} + \frac{d}{r^3}$, *i.e.* $a + D_1$.

The integral part a gives us the first required digit.

$$D_1 r = \left(\frac{b}{r} + \frac{c}{r^2} + \frac{d}{r^3}\right) r = b + \frac{c}{r} + \frac{d}{r^2}.$$

The integral part b gives us the second required digit, and so on until the operation either comes to an end or we get as many digits as are required.

Example. Convert ·7234 from scale 10 into scale 6.

$$\begin{array}{r} \cdot 7234 \\ 6 \\ \hline 4 \cdot 3404 \\ 6 \\ \hline 2 \cdot 0424 \\ 6 \\ \hline 0 \cdot 2544 \\ 6 \\ \hline 1 \cdot 5264 \end{array} \qquad \text{Result } \cdot 4201 \ldots$$

Example. How may a mass of 236 lbs. be weighed with no weights except 1 lb., 2 lbs., 2^2 lbs., 2^3 lbs., etc., only one of each sort being used. This question can be solved by converting 236 into the scale of 2.

$$\begin{array}{r|l} 2 & 236 \\ 2 & 118 \ldots 0 \\ 2 & 59 \ldots 0 \\ 2 & 29 \ldots 1 \\ 2 & 14 \ldots 1 \\ 2 & 7 \ldots 0 \\ 2 & 3 \ldots 1 \\ & 1 \ldots 1 \end{array}$$

From this it is clear that $236 = 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 1 \times 2^3 + 1 \times 2^2$.

$\therefore$ the weights used are 2^7, 2^6, 2^5, 2^3, 2^2.

294. *If a number* N *in the scale of* r *be divided by* $r-1$ *the remainder is the same as that which arises from dividing the sum of the digits of* N *by* $r-1$.

Let the digits of N be a_n, a_{n-1}, ... a_1, a_0 being the units digit.

$$\begin{aligned} \mathsf{N} &= a_n r^n + a_{n-1} r^{n-1} + \ldots + a_2 r^2 + a_1 r + a_0 \\ &= a_n (r^n - 1) + a_{n-1}(r^{n-1} - 1) + \ldots + a_2(r^2 - 1) + a_1(r-1) \\ &\qquad + a_n + a_{n-1} + \ldots + a_2 + a_1 + a_0. \end{aligned}$$

But $r^n - 1$, $r^{n-1} - 1$, etc., are divisible by $r - 1$;

$$\therefore \frac{\mathsf{N}}{r-1} = \text{an integer} + \frac{\text{the sum of the digits}}{r-1}.$$

$\therefore$ the only remainder is that which arises from dividing the sum of the digits by $r - 1$.

An important use is made of this in testing the results of multiplication in the common scale.

The remainder when a number is divided by 9 is at once got by adding up the digits of the number and seeing what remainder is found after division by 9.

Thus it is easy to express multiplier and multiplicand in the form $9m + a$ and $9n + b$.

The product is $81mn + 9(an + bm) + ab$.

$\therefore$ the product when divided by 9 gives the same remainder as ab divided by 9.

This testing is called **casting out the nines.**

Example.

$$\begin{array}{r l}
437 & \ldots \text{ the 'nine-over' is } 5 \\
\underline{618} & \ldots\ldots\ldots\ldots\ldots\ldots\ldots 6 \\
2622 & \\
427 & \\
\underline{3496} & \\
\underline{269966} & . \text{ the 'nine-over' should be } 3.
\end{array}$$

Suppose it is required to test the correctness of this multiplication.

In the multiplicand the remainder after dividing by nine, or the 'nine-over,' is 5; in the multiplier it is 6; therefore in the product it ought to be 30, or rather 3 (since $30 \div 9$ gives remainder 3). It is seen to be 2 on examination of the product; therefore the result is wrong. The error will be found in the 2nd line of multiplication.

Examples. XLVII.

Convert the following:

1. 432 from scale 10 to scale 7.
2. 678 from scale 10 to scale 6.
3. 11111 .... 9.
4. 505 .... 11.
5. 1739 .... 12.
6. ·568 .... 5.
7. ·361 .... 8.
8. ·95 .... 6.
9. 23·125 .... 4.
10. 3050 from scale 6 to scale 10.
11. 100*e* from scale 12 to scale 10.
12. 5324 .... 7 .... 9.
13. 647 .... 8 .... 6.
14. Multiply 2*t*1 by 38 in scale 11.

15 Convert 2*t*1 and 38 from scale 11 to scale 10, multiply the results together and convert to scale 11.

16. Express the sum of series $2+5\times 7+6\times 7^2+3\times 7^3$ as a number in the scale of 7, and hence find its value in the ordinary scale.

17. The number 11111 is in the scale of 5. Express it in full as a series and find its sum (1) as a G.P., (2) by converting 11111 into the scale of 10.

18. Express 2157 in scale 8.
19. Express 16935 in scale 7.

20. .... 471664 in scale 12.

21. .. .. 62·48 in scale 5, and 16·935 in scale 7.

22. .... ·5 in scale 7, and prove the result by summing a G.P.

23. Transform 101211 from scale 9 to scale 7.

24. .... 122·2 .. .. 4 .... 10.

25. .... 26·3 .... 12 ... . 10.

26. .... 9294 .... 12 .... 9.

27. .... *t*7*e*9*ee* .... 12 .... 10.

28. Multiply 26·3 by 16·7 in scale 12.

29. Divide 252710 by 249 in scale 12. Test the result by reducing dividend, divisor and quotient to scale 10.

30. Find the sq. rt. of 223551 in scale 8; prove by multiplication.

31. Find the sq. rt. of 365·738 in scale 9 to 3 places.

32. In what scale is 4072 expressed by 30504?

33. In what scale is 564 expressed by 686?

34. Prove that in any scale 10404 is a square. If the square root be 51 in scale 10, what is the scale in which 10404 is expressed?

35. What number of 2 digits is expressed by the same digits in the scales of 5 and 7?

36. Show how to weigh 85 lbs. using only weights 1, 2, 2^2, 2^3, ... lbs.

37. Weigh 95 lbs. with weights 1, 3, 3^2, ... lbs. Weigh it also using not more than one of each sort of weight, but in either scale.

38. Weigh 304 lbs. using the weights 1, 3, 3^2, ... lbs., not more than one of each, but in either scale.

39. Test the correctness of the following by "casting out the nines":

$$674 \times 504 = 339596. \qquad 743 \times 2957 = 2197051.$$
$$333^2 = 110789. \qquad 37246 \times 117 = 3972782.$$
$$314^2 = 98576.$$

What errors will this method not detect?

40. In a scale of radix r the difference between any number and another containing the same digits is divisible by $r - 1$.

CHAPTER XLVIII.

PERMUTATIONS AND COMBINATIONS.

295. If there are a number of objects which we can select and arrange in different orders, the various orders of them are called **Permutations.**

Thus 4 letters, a, b, c, d could be placed as follows:

abcd, *abdc*, *acbd*, *acdb*, *adbc*, *adcb*,
bacd, *badc*, *bcad*, *bcda*, *bdac*, *bdca*,
cabd, *cadb*, *cbad*, *cbda*, *cdab*, *cdba*,
dabc, *dacb*, *dbca*, *dbac*, *dcab*, *dcba*.

Thus there are 24 different permutations in each of which all the 4 letters appear.

If we took only 3 letters, say a, b, c, there would be 6 permutations, namely,

abc, *acb*, *bac*, *bca*, *cab*, *cba*.

On the other hand, if we were to form *groups* of letters (or other objects), without paying any attention to the order in those groups, we should be forming **Combinations.**

For instance, if from 4 letters, a, b, c, d, we were to select different groups of 3 letters, we should only get 3 groups, namely,

$$abc, \quad acd, \quad abd, \quad bcd.$$

This is expressed by saying that the number of Combinations of 4 things taken 3 at a time is 4.

The number of Combinations of n things taken r at a time is denoted by ${}^n\mathrm{C}_r$.

The number of Permutations of n things taken r at a time is denoted by ${}^n\mathrm{P}_r$.

$\therefore$ the above results may be summed up as follows:

$${}^4\mathrm{P}_4 = 24, \quad {}^3\mathrm{P}_3 = 6, \quad {}^4\mathrm{C}_3 = 4.$$

It may be observed ${}^4\mathrm{C}_1 = 1$; for out of 4 letters we can only make *one* group of 4.

The Combinations of 4 letters taken 3 at a time are 4 in number and are given above, viz.:

$$abc, \quad acd, \quad abd, \quad bcd.$$

We have noticed also that one of these groups, abc, can be arranged in 6 different orders.

So also can acd; and so also can abd and bcd.

These different orders are Permutations.

Thus the number of Permutations of 4 objects 3 at a time is 6 times the number of Combinations,

$$i.e.\ {}^4\mathrm{P}_3 = 6 \,.\, {}^4\mathrm{C}_3 = 24.$$

DEF. The different groups of r things chosen out of n things without reference to order are called the **Combinations** of n things r at a time.

Each group can be called an **r-Combination** of n things.

DEF. The different ways in which r things can be selected from n things, regard being paid to the order of selection or arrangement, are called the **Permutations** of n things r at a time.

Thus ab and ba are not different Combinations, but they are different Permutations.

296. Important. If there are 2 ways of performing one operation and 3 ways of performing another, the number of ways of performing the two operations in connection with each other is 2×3, *i.e.* 6; for each of the 2 ways can be taken with each of the 3 ways.

In more general terms, if there are x ways of performing one operation, and y ways of performing another, there are xy ways of performing the two operations combined.

For instance, if there are x ways of selecting an orange and y ways of selecting an apple, the number of ways of selecting both an orange and an apple is xy.

297. Without actually writing down all the different permutations of a group of 3 letters a, b, c, we might find out as follows how many there could be.

Imagine that they have to be put into certain empty spaces, 3 in number.

The 1st space can be filled in 3 different ways, viz., with a, b, or c.

When the 1st space has been filled there remain 2 letters for filling the 2nd space. For if a were chosen for the 1st space, there would be b and c left; if b were chosen for the 1st space, there would be a and c left.

Consequently there are 2 ways of filling the 2nd space.

$\therefore$ the number of ways of filling the first two spaces is 3×2.

There is only one way of filling the last space, for there is only one letter for it.

Thus the number of ways of filling all 3 spaces is $3 \times 2 \times 1$, *i.e.* 6.

$\therefore$ the number of arrangements of a group of 3 things (or, in other words, the number of permutations of 3 things taken all together) is $3 \times 2 \times 1$.

[This is the number that we found by writing them all down.]

Similarly, if there were 4 things to be arranged in order, we could fill the 1st place in 4 ways.

There would then be left a choice of 3 things.

$\therefore$ we could fill the 2nd place in 3 ways.

$\therefore$ the first 2 places could be filled in 4×3 ways.

There would be 2 ways of filling the 3rd place.

$\therefore$ the number of ways of filling the first 3 places is $4 \times 3 \times 2$.

There would be only one way of filling the last place.

$\therefore$ the number of ways of filling all the places is $4 \times 3 \times 2 \times 1$, *i.e.* 24.

298. *To find the value of* ${}^5\mathrm{P}_2$.

Imagine that we have two empty spaces to fill.

The 1st place can be filled with any of the 5 things.

$\therefore$ there are 5 ways of doing this.

For the 2nd place we have 4 remaining for selection.

$\therefore$ the 2nd place can be filled in 4 ways.

$\therefore$ the number of ways in which the 2 places can be filled is 5×4, *i.e.* 20.

This sort of question differs from the previous one; for here we are taking the 5 things not all together, but 2 at a time.

299. *To find the number of permutations of n things taken r together.*

Denote the n things by n letters a, b, c, etc.

Consider that there are r places to be filled, each with one letter.

Any one of the n letters can be chosen to fill the first place.

$\therefore$ there are n ways of filling the first place.

Whichever letter is chosen, it may be followed by any one of the remaining $n-1$ letters.

$\therefore$ there are $n-1$ ways of filling the second place.

$\therefore$ the number of ways in which the first two places can be filled is $n(n-1)$, *i.e.* ${}^n\mathrm{P}_2 = n(n-1)$. (Art. 296)

There are $n-2$ ways of filling the third place.

$\therefore$ the number of ways of filling the first three places is

$$n(n-1)(n-2),$$

$$i.e.\ {}^n\mathrm{P}_3 = n(n-1)(n-2).$$

Continuing this process we see that ${}^n\mathrm{P}_r$ contains r factors, and thus

$$\begin{aligned} {}^n\mathrm{P}_r &= n(n-1)(n-2)\ldots \text{ continued to } r \text{ factors} \\ &= n(n-1)(n-2)\ldots(n-r+1). \end{aligned}$$

COROLLARY. If we continue this process as far as n factors, the last factor is 1.

Thus $$^nP_n = n(n-1)(n-2)\ldots 1.$$

This product (of all integers from 1 to n) is denoted by $\underline{|n}$ or $n!$, and is called **factorial n.**

Example. How many numbers lying between 3000 and 4000 and divisible by 5 can be made with the digits 3, 4, 5, 6, 7, 8?

This is evidently a case of permutations, not combinations; for the order is important.

As the number is between 3000 and 4000, it must begin with 3, and it must contain 4 digits. Also, since it is divisible by 5, it must end with 5 (since there is no 0).

Thus the number has its first and last digits given, and it only remains to fill up the 2 middle places out of the digits 4, 6, 7, 8.

The number of ways of doing this is $^4P_2$, *i.e.* 12.

300. *To find the number of* **r-combinations** *of n things, i.e. the number of combinations of n things taken r together.*

As we have seen, each combination containing r things can be arranged in $\underline{|r}$ different ways; ($^rP_r = \underline{|r}$), *i.e.* each combination produces $\underline{|r}$ permutations.

$$\therefore\ {}^nP_r = \underline{|r}\,.\,{}^nC_r;$$

$$\therefore\ {}^nC_r = \frac{^nP_r}{\underline{|r}} = \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\ldots\overline{n-r+1}}{\underline{|r}}.$$

[r factors above and below.]

This may be put entirely in terms of factorials by multiplying numerator and denominator by the product of all the consecutive integers below $n-r+1$, *i.e.* by $(n-r)(n-r-1)\ldots 1$.

$$\text{Thus}\quad {}^nC_r = \frac{n\,.\,\overline{n-1}\,.\,\overline{n-2}\ldots\overline{n-r+1}\,.\,\overline{n-r}\,.\,\overline{n-r-1}\ldots 1}{\underline{|r}\,.\,\overline{n-r}\,.\,\overline{n-r-1}\ldots 1}$$

$$= \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}$$

301. *The number of combinations of n things taken n − r at a time = the number when taken r at a time.*

$$^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

By substituting $n-r$ for r, we get

$$^nC_{n-r}=\frac{\lfloor n}{\lfloor n-r\,\lfloor n-(n-r)}=\frac{\lfloor n}{\lfloor n-r\,\lfloor r}.$$

$$\therefore\ {}^nC_{n-r}={}^nC_r.$$

This fact is also evident from the consideration that every choice of r things leaves behind $n-r$ things.

Thus the number of ways of forming a group of r things is the number of ways of forming another group (a *complementary* group) of $n-r$ things, *i.e.* $^nC_r={}^nC_{n-r}$.

302. It should be noticed that, if it is desired to find the number of r-combinations in each of which a specified object occurs, it is simply necessary to put aside the specified object in order to join it to each of the combinations, $r-1$ together, that can be formed out of the other $n-1$ objects.

$$\therefore\ \text{the required number is } {}^{n-1}C_{r-1}.$$

Example 1. A picket of 6 has to be formed out of 10 men. How many different selections can be made?

The number $={}^{10}C_6={}^{10}C_{10-6}={}^{10}C_4=\frac{10.9.8.7}{1.2.3.4}=210.$

In how many of these selections will one specified man be included?

Put him on one side; form pickets of 5 from the remaining 9 men, and then add him to each picket.

The number of pickets thus formed $={}^9C_5={}^9C_4$

$$=\frac{9.8.7.6}{1.2.3.4}=126.$$

How many selections can be made so as not to contain one specified man?

Leave him out, and form pickets of 6 from the remaining 9.

The number of pickets $={}^9C_6={}^9C_3=\frac{9.8.7}{1.2.3}=84.$

Example 2. Out of a town council of 16 members, of whom 9 are Conservatives and 7 Liberals, in how many ways can a committee of 11 be chosen so as to contain 6 Conservatives and 5 Liberals?

The number of different groups of 6 Conservatives which can be selected out of the $9={}^9C_6={}^9C_3=\frac{9.8.7}{1.2.3}=84.$

The number of Liberal groups of $5={}^7C_5={}^7C_2=\frac{7.6}{1.2}=21.$

Each of the 21 groups can be combined with each of the 84 groups to form a committee.

$\therefore$ the number of different committees

$$=84\times 21=1764.$$

Examples. XLVIII. a.

1. Put down all the groups of 2 letters which can be formed out of the 4 letters a, b, c, d. How many are there?

2. Put down all the arrangements of 4 of the 5 letters a, b, c, d, e which can be made with a standing first in each. How many are there?

3. Put down all the arrangements of the 5 letters a, b, c, d, e such that each begins with a and ends with e, or *vice versa*. How many are there?

4. How many factors are there in $n \cdot \overline{n-1} \cdot \overline{n-2} \ldots \overline{n-6}$?

5. .... $n \cdot \overline{n-1} \cdot \overline{n-2} \ldots \overline{n-7}$?

6. In $\overline{n-1} \cdot \overline{n-2} \cdot \overline{n-3} \ldots$ to r factors, what is the last one?

7. Simplify $^nP_r \div {}^{n-1}P_r$.

8. In $^{18}P_{15} \div \underline{|15}$ how many factors are alike in numerator and denominator? Give the result of cancelling them.

9. Find the numerical value of $^7P_4$.

10. .... $^9P_3$.

11. .... $^{10}P_5$.

12. How many changes can be rung on 7 bells, using all?

13. If $^nP_3 : {}^nP_4 = 1 : 7$, find n.

14. $^nP_3 = 20n$; find n.

15. If $^nP_7 = 12\,{}^nP_6$, find n.

16. Prove that $^{n+1}P_r = {}^nP_r + r \cdot {}^nP_{r-1}$.

17. If $^nP_4 = 840$, find n. (Let $n^2 - 3n = x$.)

18. Find the numerical value of $^9C_3$ and of $^9C_6$.

19. .... $^8C_5$.

20. .... $^{12}C_{10}$.

21. Three travellers arrive at an inn and find that there are 7 vacant bedrooms. In how many ways can they be assigned rooms, one to each?

22. Two men enter a railway carriage in which there are 8 vacant seats. In how many ways can they be seated?

23. How many different groups of 3 cards can be chosen out of 13?

24. A subscriber to a library is allowed to take out 2 books. In how many ways can he select 2 out of 50?

25. The number of permutations of a certain number of things 3 at a time is 20 times the number of permutations of half the number of things taken 2 at a time. How many things are there?

26. In how many ways can 5 people each choose one room out of 8 rooms?

27. If $^nP_4 = 6 \cdot {}^nP_3$, find n.

28. Four men are chosen by lot out of 10. In how many cases is one particular man chosen?

29. If ${}^nP_r = 120\,{}^nC_r$, find r.

30. If ${}^{15}C_r = {}^{15}C_6$, find r.

31. Find ${}^{10}C_4$ and ${}^{10}C_3$. Add the results and prove that ${}^{11}C_4$ is obtained. State and prove the general theorem of which this is a particular instance.

32. Out of the 7 common silver coins, from the crown to the threepenny piece, how many different selections of 5 coins are possible?

33. Out of 7 men, how many crews can be chosen to row in a four-oared boat? How many, if one of the 7 has been already selected for stroke?

34. Out of 8 men and 6 women, how many different committees can be formed composed of 5 men and 4 women?

35. There are n points in a plane, no three of which are in a straight line. Find how many straight lines can be formed by joining them.

36. How many numbers can be formed with three digits out of the digits 1, 2, 3, 4, 5?

In how many of these will the digit 1 stand first?

37. How many numbers can be formed each containing all the digits 0, 1, 2, 3, 4?

38. How many numbers of three digits can be formed from the digits 0, 1, 2, 3, 4, 5?

303. *To find the number of* **r-combinations** *of n things without assuming the number of permutations.*

Denote the things by a, b, c, etc.

The number of combinations in which a occurs is found by combining the remaining $n-1$ letters $r-1$ together, and is therefore ${}^{n-1}C_{r-1}$. (Art. 302.)

Similarly for each of the n letters.

$\therefore$ the total number of r-combinations thus obtained

$$= n \,.\, {}^{n-1}C_{r-1}.$$

But in these we shall find each combination occurring r times. *e.g.* $abcd\ldots$ (consisting of r letters) will be found amongst those containing a, amongst those containing b, c, and so on.

$\therefore$ the total number of r-combinations thus obtained

$$= r \,.\, {}^nC_r.$$

$$\therefore\ r \,.\, {}^nC_r = n \,.\, {}^{n-1}C_{r-1}. \qquad \ldots(1)$$

$${}^nC_r = \frac{n}{r} \,.\, {}^{n-1}C_{r-1}. \qquad \ldots(2)$$

Similarly, $${}^{n-1}C_{r-1} = \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

and $${}^{n-2}C_{r-2} = \frac{n-2}{r-2} \cdot {}^{n-3}C_{r-3},$$

and so on.

$$\therefore\ {}^nC_r = \frac{n}{r} \cdot {}^{n-1}C_{r-1} = \frac{n}{r} \cdot \frac{n-1}{r-1} \cdot {}^{n-2}C_{r-2}$$
$$= \ldots\ldots$$
$$= \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \ldots \overline{n-r+2}}{r \cdot \overline{r-1} \cdot \overline{r-2} \ldots 2} \cdot {}^{n-r+1}C_1$$
$$= \frac{n \cdot \overline{n-1} \cdot \overline{n-2} \ldots \overline{n-r+1}}{\lfloor r}.$$

Multiply by $\frac{\lfloor n-r}{\lfloor n-r}$; $\quad \therefore\ {}^nC_r = \frac{\lfloor n}{\lfloor r \lfloor n-r}$.

304. *To prove* ${}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

${}^nC_{r-1}$ is the number of combinations of $n+1$ things r together which contain a specified thing. (See Art. 302.)

${}^nC_r$ is the number of combinations of $n+1$ things r together which do not contain that specified thing. (See Art. 302, Ex. 1.)

The sum of these two must be the total number of combinations of $n+1$ things r together;

$$\therefore\ {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r.$$

As an exercise prove this from the value of ${}^nC_r$. (Art. 316.)

305. *To find for what value of r ${}^nC_r$ is greatest, supposing n given.*

$${}^nC_{r+1} = \frac{n(n-1)\ldots(n-r+1)(n-r)}{1 \cdot 2 \ldots r \cdot (r+1)};$$

$${}^nC_r = \frac{n(n-1)\ldots(n-r+1)}{1 \cdot 2 \ldots r};$$

$$\therefore\ \frac{{}^nC_{r+1}}{{}^nC_r} = \frac{n-r}{r+1};$$

$$\therefore\ {}^nC_{r+1} = \frac{n-r}{r+1}\,{}^nC_r.$$

Now $\frac{n-r}{r+1}\left[\text{which } = \frac{n+1-(r+1)}{r+1} = \frac{n+1}{r+1} - 1\right]$ diminishes as r increases, but is at first > 1.

$$\therefore \quad {}^nC_{r+1} > {}^nC_r \text{ as long as } \frac{n-r}{r+1} > 1;$$

$$\therefore \quad {}^nC_r \text{ has its greatest value when } \frac{n-r}{r+1} = 1 \quad \text{or is first} < 1,$$

$$\text{i.e. when } n - r = r + 1 \quad \ldots\ldots\ldots \quad < r+1,$$

$$\ldots\ldots\ldots\ldots \; n - 1 = 2r \quad \ldots\ldots\ldots \quad < 2r,$$

$$\frac{n-1}{2} \quad\quad > \frac{n-1}{2}.$$

(i) If n is odd,

the greatest value of ${}^nC_r$ is $\quad {}^nC_{\frac{n-1}{2}} = {}^nC_{\frac{n+1}{2}} = \dfrac{\lfloor n}{\lfloor \frac{n-1}{2} \lfloor \frac{n+1}{2}}.$

(ii) If n is even,

the greatest value of ${}^nC_r$ is $\quad {}^nC_{\frac{n}{2}} = \dfrac{\lfloor n}{\lfloor \frac{n}{2} \lfloor \frac{n}{2}}.$

Example. A man has 10 friends whom he wishes to invite to dinner so that there may not be the same guests at any two dinners, the same number being invited each time. If he wishes to have as many dinner parties as possible, how many must he invite each time?

The number of parties he can form is ${}^{10}C_r$, where r is the number invited each time.

$\therefore$ the question is: 'For what value of r is ${}^{10}C_r$ greatest?'

$${}^{10}C_{r+1} = \frac{10.9.8\ldots(11-r)(10-r)}{1.2\ldots(r+1)}.$$

$${}^{10}C_r = \frac{10.9.8\ldots(11-r)}{1.2\ldots r}.$$

$$\therefore \quad \frac{{}^{10}C_{r+1}}{{}^{10}C_r} = \frac{10-r}{r+1}.$$

Now $\dfrac{10-r}{r+1}$ diminishes as r increases, but is at first >1.

$$\therefore \quad {}^{10}C_{r+1} > {}^{10}C_r \text{ as long as } \frac{10-r}{r+1} > 1.$$

$$\therefore \quad {}^{10}C_r \text{ has its greatest value when } \frac{10-r}{r+1} = 1 \quad \text{or is first} < 1,$$

$$\text{i.e. when } 10 - r = r + 1 \;\ldots\ldots\ldots\ldots\; < r+1,$$

$$\ldots\ldots\ldots\ldots \quad 9 = 2r \quad \ldots\ldots\ldots\ldots \; < 2r,$$

$$\text{i.e. when } r \text{ is first} > 4\tfrac{1}{2},$$

$$\text{i.e. when } r = 5;$$

i.e. the number of guests each time should be 5.

306. *To find the number of permutations taken all together of n things which are not all different.*

Suppose them to be n letters, of which p letters are a, q letters are b, r letters are c, and the rest are all different.

Let P be the required number of permutations.

If the p letters a were changed into p fresh letters, different from one another and from the rest, the number of permutations would be thereby multiplied by $p!$, since the group of p fresh letters would admit of $p!$ arrangements.

Similarly in the case of the q letters b and the r letters c.

$\therefore$ the total number of permutations would then be

$$P \cdot p! \cdot q! \cdot r!.$$

But there would then be n letters all different, and consequently the number of permutations of them taken all together would be $n!$.

$$\therefore\ P \cdot p! \cdot q! \cdot r! = n!;$$

$$\therefore\ P = \frac{n!}{p!\,q!\,r!}.$$

Example. Find the number of permutations of the letters of **mammal** taken all together.

There are 6 letters, of which 3 are m's, 2 are a's.

$$\text{The number of permutations} = \frac{6!}{3!\,2!} = \frac{6.5.4.3.2}{3.2.2}$$
$$= 60.$$

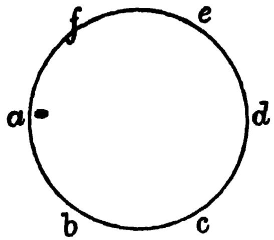
Permutations in a Ring.

307. If *abcdef* and *fabcde* represent objects arranged (in each case) in a straight line, no one will hesitate to pronounce these two to be different arrangements.

If they are placed on the circumference of a circle, they can hardly be pronounced to be different. The circular order is the same for both, as is obvious from a figure.

Suppose that the objects are persons seated at a round table. From *abcdef* to *fabcde* the change is only apparent. There has

been no change in the order, only a movement of the whole one place round the table.



To prevent this movement, which gives no fresh order, we must suppose one person to be fixed, and find the number of permutations of the rest.

For example, if there are 6 persons to be seated at a round table, let one person take his seat. The remaining 5 can be arranged in $\underline{|5}$ different ways.

308. *Beads in a necklace.*

If n different beads were to be arranged to form a necklace, the number of different orders in which they could be put would be $\frac{1}{2}\underline{|n-1}$.

For consider what happens when a necklace of 6 beads, arranged in the order *abcdef*, is turned over. The arrangement, *read the same way round as before*, becomes *fedcba*.

Thus what were two different arrangements in the case of persons in a ring become identical in a necklace.

$\therefore$ the number of arrangements of n beads in a necklace = half the number of arrangements of n persons at a round table $= \frac{1}{2}\underline{|n-1}$.

Example. How many arrangements of 4 men and 4 women at a round table can be made, if the men and women are to be alternate?

Place the 4 men round the table with alternate places left vacant. The number of ways of doing it is $\underline{|3}$.

The 4 vacant places can be filled by the women in $\underline{|4}$ different ways.

$\therefore$ the total number of arrangements

$$=\underline{|3}\times\underline{|4}=6\times 24=144.$$

XLVIII. b.

1. Find the number of arrangements of the letters of *Division.*

2. .... *Sirocco.*

3. .... *Mammalia.*

4. In how many ways can 7 persons be arranged in a ring?

5. Find the number of arrangements of the letters of *Onoto.*

6. .... *Assassin.*

7. In how many ways can 5 persons be arranged in a ring?

8. In how many ways can 7 different beads be arranged on a string to form a necklace?

9. In how many ways can 6 persons be arranged at a round table?

10. In how many ways can a committee of n persons be seated round a table, if the chairman must always be opposite to the fireplace?

11. Five gentlemen and five ladies are to be seated, ladies and gentlemen alternately, at a round table. In how many ways can it be done?

12. There are 7 letters, of which some are a's, and the rest different. The number of different words that can be made with them, taking them all together, is 210. How many are a's?

13. Find the greatest value of ${}^9C_r$.

14. A man makes different selections of r things out of 14 things. If the number of selections is to be the greatest possible, what must be the value of r?

15. How many triangles can be formed with 9 straight lines, of which no two are parallel, and no three are concurrent?

16. From 8 men and 9 women a committee is to be formed, comprising 5 of each sex. In how many ways can it be done?

17. How many different sets for lawn-tennis can be formed from 10 ladies and 6 gentlemen, each set containing 2 ladies and 2 gentlemen, the ladies not playing on the same side?

309. Questions sometimes occur in which it is required to find the number of ways of dividing up n things into parcels.

The method of solution follows at once from the formula for ${}^nC_r$.

Suppose that it is required to divide n things into two parcels, one containing r things, the other containing the rest.

The number of ways of choosing r things for the first parcel is ${}^nC_r$, and the remaining things form the second parcel.

Thus the number of ways of forming the two parcels is the same as the number of ways of choosing the contents which make up the first parcel.

$$\therefore \text{ the number of ways} = {}^nC_r = \frac{\underline{|n}}{\underline{|r}\,\underline{|n-r}}.$$

If $m+n$ things have to be divided into two parcels of m things and n things respectively, the number of ways of doing this

$$= {}^{m+n}C_m = \frac{\underline{|m+n}}{\underline{|m}\,\underline{|n}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If $m+n+p$ things have to be divided into three parcels of m things, n things, and p things respectively, we first consider them divided into two parcels containing m things and $n+p$ things respectively.

From (1) we see that the number of ways in which this can be done is $\dfrac{\underline{|m+n+p}}{\underline{|m}\,\underline{|n+p}}$.

Similarly the number of ways of dividing $n+p$ things into two parcels containing n and p things respectively is $\dfrac{\underline{|n+p}}{\underline{|n}\,\underline{|p}}$.

$\therefore$ the number of ways of dividing $m+n+p$ things into three parcels of m, n, and p things respectively

$$= \frac{\underline{|m+n+p}}{\underline{|m}\,\underline{|n+p}} \cdot \frac{\underline{|n+p}}{\underline{|n}\,\underline{|p}} = \frac{\underline{|m+n+p}}{\underline{|m}\,\underline{|n}\,\underline{|p}},$$

and so on for a larger number of parcels.

The case where $m=n$ (in the distribution of $m+n$ things into two parcels) has to be considered.

Here the formula $\dfrac{\underline{|m+n}}{\underline{|m}\,\underline{|n}}$ will not give the correct answer.

For consider the simple case of 4 things a, b, c, d distributed into parcels of two and two things.

We have

ab	and	cd,
ac	......	bd,
ad	......	bc,
bc	......	ad,
bd	......	ac,
cd	......	ab.

As the formula tells us, we have $\frac{\lfloor 4}{\lfloor 2 \lfloor 2}$, *i.e.* 6 ways of doing it: *but each way occurs twice.*

$\therefore$ the number of different ways of doing it is only half the above.

Similarly, when $2m$ things are to be divided into 2 parcels each containing m things, the number of ways of doing it is

$$\frac{\lfloor 2m}{2\lfloor m \lfloor m}.$$

If these parcels were to be given to 2 persons, the number of ways of doing it would be $\frac{\lfloor 2m}{2\lfloor m \lfloor m} \times 2$, *i.e.* $\frac{\lfloor 2m}{(\lfloor m)^2}$; for after making up the $2m$ things into any two parcels, you could give these two parcels in two ways to the persons

Similarly, in the formula $\frac{\lfloor m+n+p}{\lfloor m \lfloor n \lfloor p}$, if $m=n=p$, and if there be no distinction between the order of the parcels, the $\lfloor 3$ different ways of arranging the parcels of m, n, and p things would become identical.

$\therefore$ the number of ways of dividing $3m$ things into 3 parcels each of m things

$$= \frac{\lfloor 3m}{\lfloor 3\,(\lfloor m)^3}.$$

If they had to be given to 3 persons, the number of ways would be $\lfloor 3$ times this; for when the parcels have been formed, the assigning of them to the 3 persons can be done in $\lfloor 3$ ways.

$\therefore$ the number of ways of providing 3 persons with m things each is

$$\frac{\lfloor 3m}{(\lfloor m)^3}.$$

Example 1. In how many ways can 10 letters be made up into 2 parcels of 6 and 4 respectively?

The number of ways is $\frac{\lfloor 10}{\lfloor 6 \lfloor 4}$, *i.e.* 210.

In how many ways can 10 letters be left at 2 houses, 6 at one, 4 at the other?

The 6 letters can be left at the first house, the 4 at the other, or *vice versa.*

$$\therefore \text{ the number of ways is } \frac{2\,\underline{|10}}{\underline{|6}\,\underline{|4}}, \text{ i.e. } 420.$$

Example 2. In how many ways can 15 letters be divided into three parcels, each containing 5?

$$\text{The number of ways} = \frac{\underline{|15}}{\underline{|5}\,\underline{|5}\,\underline{|5}} \times \frac{1}{\underline{|3}}.$$

In how many ways can 15 letters be left at 3 houses, 5 at each?

The number of ways $= \underline{|3}$ times the number of ways of making up the parcels of $5 = \dfrac{\underline{|15}}{(\underline{|5})^3}$.

310. Interesting applications of the theory of Combinations occur in geometry; *e.g.* the enumeration of the diagonals of a given polygon. The following are instances of such questions.

Example 1. Find the number of diagonals of an octagon.

The 8 vertices are to be taken two together, and connected by straight lines.

The number of connecting lines $= {}^8C_2 = \dfrac{8 \cdot 7}{1 \cdot 2} = 28$.

But this includes the 8 sides of the octagon.

$\therefore$ the number of diagonals is 20.

Example 2. Find the number of regions into which a plane is divided by n straight lines, of which no two are parallel and no three meet in a point.

When there is no straight line, the plane forms one region. The introduction of a straight line divides it into 2 regions.

The addition of a second line makes 4 regions in all,

i.e. the 2nd line adds on 2 regions, making $1+1+2$ regions,

the 3rd line adds on 3 regions, making $1+1+2+3$ regions,

and so on.

Thus the total number of regions, when there are n lines,

$$= 1 + 1 + 2 + 3 + \ldots + n$$

$$= 1 + \frac{n(n+1)}{2} = \frac{1}{2}(n^2 + n + 2).$$

311. *To find the number of combinations of n things taken any number at a time.*

There are 2 ways of dealing with each of the n things; we may take it or leave it.

As each can be dealt with in 2 ways, the number of ways of treating the n things is $2 . 2 . 2 \ldots$, *i.e.* 2^n.

But this would include the case in which all are rejected.

$\therefore$ the total number of combinations of n things taken *any* number at a time $= 2^n - 1$.

XLVIII. c.

1. Prove that ${}^nP_r = {}^{n-1}P_r + r \,.\, {}^{n-1}P_{r-1}$.

2. In the combinations of 9 things, 3 at a time, prove that any particular thing occurs in one-third of the whole number of combinations.

3. If $n : {}^nP_3 = 1 : 72$, find n.

4. Prove that ${}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6 = 2^6 - 1$.

5. Find the number of ways of choosing one or more (up to 7) things out of 7.

6. How many different sums can be paid by a man who has in his pocket the following coins: a farthing, a penny, a sixpence, a shilling, a half-crown, a crown, a half-sovereign, and a sovereign?

7. A person who chooses 3 cards out of a heap of cards has 425 times as many possible selections as one who only chooses a single card. How many cards are there in the heap?

8. If ${}^nP_3 : {}^nC_5 = 1 : 5$, find n.

9. How many numbers can be formed with the digits 3, 4, 5, each digit occurring once in each number? What is the sum of all such numbers?

10. How many ways are there of making up an eleven out of 15 boys, if there are to be just 4 bowlers in it, and only 5 of the 15 are bowlers?

11. From 13 members, of whom only 4 can bowl, how many different elevens can be made so as to include 2 bowlers *at least*?

12. A person has 8 friends, among whom he wishes to make up as many dinner parties as possible, inviting the same number each time. How many times is one particular person invited?

13. In how many ways can a party of 5 men be chosen from 10 men? In how many ways if (1) one man is always to be included, (2) if one man is always to be excluded?

14. A committee of 7 is to be formed from 10 Conservatives and 8 Liberals, so that there may be 4 Conservatives and 3 Liberals on it. How many different committees can be formed?

15. Out of 17 consonants and 5 vowels how many words can be formed containing 2 consonants and 1 vowel?

16. How many words of 11 letters may be formed of the letters in $a^3b^5c^2d$?

17. How many words may be formed of all the letters of *accommodation*?

18. How many straight lines can be made by joining 4 points, no 3 of which are collinear?

19. If $y = {}^8C_x$, plot the positions of the point (x, y) as x takes successive integral values. Find what value of x gives a maximum value to y.

(Take the unit for y much greater than that for x.)

20. In how many ways can a crew of 8 men be arranged in a boat? If only 3 are fit to row stroke, in how many ways can they then be arranged?

21. How many numbers can be formed of the digits 1, 2, 3, 4, each digit occurring once in a number? Find the sum of all such numbers.

22. A party is chosen from 18 men. The number of ways of choosing it is greater than it would be for a party of a different size. Of how many does it consist? How many of these selections contain 3 given men?

23. In how many ways can I choose one or more out of 8 books?

24. Out of 9 objects, in how many ways can I choose *at least* 2?

25. Find the number of permutations of 10 things 7 at a time, in which 3 particular things occur.

26. If ${}^{n+1}C_{r+2} = 9 \times {}^nC_{r+1} = 90 \times {}^{n-1}C_r$, find n and r.

27. How many different numbers may be composed with the digits of 111223, each consisting of 6 digits?

28. In how many ways can 7 sovereigns and 5 shillings be given to 12 men, one coin to each?

29. Out of 6 letters, of which some are a's and the others all different, 120 different words can be formed. How many a's are there?

30. Twenty-six people are to travel by an omnibus which can carry 12 inside and 14 outside. If 8 of them will not ride outside and 6 will not go inside, in how many different ways can the party travel, without regard to the *order* of seating?

31. Of 15 men 10 can row and cannot steer, and 5 can steer and cannot row; find how many crews of 8 rowers and a coxswain can be formed out of the 15 men?

32. Find the number of diagonals of a decagon.

33. ... . dodecagon.

34. A committee of 7 has to be chosen out of 13 persons, of whom 6 are Liberals and 7 Conservatives. In how many ways can it be done so as to give a Liberal majority on the committee?

35. How many different permutations can be made of the letters of the word *essences*, using all the letters; and how many of these will begin with n and end with s?

Harder Examples. XLVIII. d.

1. Eight men have to be arranged to row in an eight-oared boat. 3 of the eight cannot row on the bow side, and 2 others cannot row on the stroke side. How many arrangements can be made?

2. You have 7 envelopes addressed to 7 people, to 4 of whom you intend to send copies of a circular, the other 3 envelopes to be used for a different purpose. In how many correct ways can you put the 4 circulars into 4 of the 7 envelopes?

3. My cousin belongs to a club of 30 members. In how many ways can I meet at most 5 of the members, of whom my cousin may or may not be one; and in how many ways can I meet 5 members including my cousin?

4. Out of a dozen persons how many parties can be made, each consisting of not more than 8?

5. Find the number of words of 2 consonants and 2 vowels formed from "education." In how many of these will the two sorts occur alternately?

6. How many triangles can be formed having their vertices at those of a given pentagon?

7. Find the number of ways in which 10 boys and 8 girls can stand in a line so that no two girls are together.

8. Fourteen different coins are contained in 2 bags, 7 in each. How many different combinations of 8 coins can be formed by taking 4 from each bag?

9. How many different numbers of 3 digits can be formed, it being allowed to use the digits 2, 3, 4, 5 once and 6 twice?

10. In how many different ways can 52 cards be dealt into 4 groups of 13 each? In how many different ways can they be dealt to 4 players, 13 to each?

11. The number of combinations of n letters, taken 5 together, in which a, b, c occur is 21. Find the number of combinations of them, taken 6 together, in which a, b, c, d occur.

12. I wish to divide 15 different postage-stamps into parcels containing 5 each. How many sets of parcels can I make?

13. There are two sets of concurrent straight lines, n in one set, p in the other; how many intersections are there if there are no parallels?

14. The number of ways in which n books can be arranged on a shelf so that 2 particular books are not together is $(n-2)\underline{|n-1}$.

15. How many Δs are formed by 2 sets of 3 str. lines, each set passing through a point, and no two lines being parallel?

16. Find the number of arrangements that can be made of the letters of the word *infinite* (1) when they are taken all together, (2) when they are taken 4 together, so that each arrangement has 2 different vowels and 2 different consonants.

17. A bag contains n sovereigns and n shillings. Prove that the number of ways in which they can be drawn out in succession, one at a time, is

$$\frac{1.3.5 \ldots (2n-1)}{\underline{|n}} \times 2^n.$$

18. In how many ways can n men be placed in a row, if two of them are forbidden to be at either end?

19. Out of n objects, how many ways are there of choosing 3 or more?

20. Find the number of diagonals of a polygon of n sides.

21. Find the number of ways of giving mn different things to n persons, so that each may have m things.

22. The number of permutations of n things r together in which x particular things occur is ${}^{n-x}P_{r-x}\,.\,{}^{r}P_{x}$.

23. In how many ways can m letters a and n letters b be arranged so that none of the b's shall be together, m being greater than n?

24. The number of ways of seating $2n$ persons at two round tables, n at each, is $\underline{|2n}/n^2$.

25. Given n points, of which only m are in one straight line, how many straight lines can be formed by joining them?

26. If m points in one straight line are joined to n points in another straight line, how many intersections are there besides the original m and n points?

27. A rectangle has p lines ruled across it parallel to one pair of sides, and p lines parallel to the other sides. Prove that the number of rectangles formed is $\frac{1}{4}(p+1)^2(p+2)^2$.

CHAPTER XLIX.

BINOMIAL THEOREM.

312. By means of the binomial theorem, any power of a binomial expression can be expressed as the sum of a number of terms.

The theorem states that

$$(\mathbf{a}+\mathbf{x})^n = \mathbf{a}^n + n\mathbf{a}^{n-1}\mathbf{x} + \frac{n(n-1)}{1.2}\mathbf{a}^{n-2}\mathbf{x}^2 + \frac{n(n-1)(n-2)}{1.2.3}\mathbf{a}^{n-3}\mathbf{x}^3 + \dots.$$

The series on the right-hand side is called the *expansion* of $(a+x)^n$.

The truth of the theorem may easily be tested in simple cases.

Thus $\qquad (a+x)^2 = a^2 + 2ax + \frac{2.1}{1.2}x^2 = a^2 + 2ax + x^2.$

$$(a+x)^3 = a^3 + 3a^2x + \frac{3.2}{1.2}ax^2 + \frac{3.2.1}{1.2.3}x^3 = a^3 + 3a^2x + 3ax^2 + x^3.$$

$$(a+x)^4 = a^4 + 4a^3x + \frac{4.3}{1.2}a^2x^2 + \frac{4.3.2}{1.2.3}ax^3 + \frac{4.3.2.1}{1.2.3.4}x^4$$
$$= a^4 + 4a^3x + 6a^2x^2 + 4ax^3 + x^4.$$

Note carefully that, in each of the above,

1. The index of the highest power of a or x is n, and its coefficient is unity.

2. The number of terms is $n+1$; *e.g.* there are 5 terms in the expansion of $(a+x)^4$.

3. The indices of a decrease by unity, and those of x increase by unity in each successive term.

4. The coefficients of terms equidistant from the beginning and the end of the series are equal.

5. Counting unity as a factor, the number of factors in the numerator of any coefficient is equal to the number of factors in its denominator.

6. The sum of the indices of a and x is n in every term. In other words, the expansion is homogeneous and of the n^{th} degree.

We shall see as we proceed that these rules hold for all values of n.

Writing $-x$ instead of x, we have

$$\begin{aligned}(\mathbf{a}-\mathbf{x})^n &= \mathbf{a}^n + \mathbf{n}\mathbf{a}^{n-1}(-\mathbf{x}) + \frac{\mathbf{n}(\mathbf{n}-1)}{1\,.\,2}\mathbf{a}^{n-2}(-\mathbf{x})^2 \\ &\quad + \frac{\mathbf{n}(\mathbf{n}-1)(\mathbf{n}-2)}{1\,.\,2\,.\,3}\mathbf{a}^{n-3}(-\mathbf{x})^3 + \dots \\ &= \mathbf{a}^n - \mathbf{n}\mathbf{a}^{n-1}\mathbf{x} + \frac{\mathbf{n}(\mathbf{n}-1)}{1\,.\,2}\mathbf{a}^{n-2}\mathbf{x}^2 \\ &\quad - \frac{\mathbf{n}(\mathbf{n}-1)(\mathbf{n}-2)}{1\,.\,2\,.\,3}\mathbf{a}^{n-3}\mathbf{x}^3 + \dots .\end{aligned}$$

313. The following is a rapid method of obtaining an expansion in simple cases.

Take $(a+x)^9$.

$$\text{The first term} = a^9; \quad \text{the second term} = 9a^8x;$$

$$\text{the third term} = \frac{9\,.\,8}{2}a^7x^2 = 36a^7x^2.$$

The coefficient of the third term is obtained from that of the second **by multiplying the index of a by the coefficient in the second term, and dividing their product by the number (2) of the term.**

Thus the fourth term $= \frac{7 \cdot 36}{3} a^6x^3 = 84a^6x^3$.

The fifth term $= \frac{6 \cdot 84}{4} a^5x^4 = 126a^5x^4$.

The sixth term $= \frac{5 \cdot 126}{5} a^4x^5 = 126a^4x^5$.

The seventh term $= \frac{4 \cdot 126}{6} a^3x^6 = 84a^3x^6$, and so on.

Thus $(a+x)^9 = a^9 + 9a^8x + 36a^7x^2 + 84a^6x^3 + 126a^5x^4$
$\qquad + 126a^4x^5 + 84a^3x^6 + 36a^2x^7 + 9ax^8 + x^9$.

314. The student who has omitted the Chapter on Permutations and Combinations should study the following.

$n(n-1)(n-2) \ldots 1$, *i.e.* the product of all integers from 1 to n, is denoted by $\underline{|n}$ or $n!$, and is called **factorial** n.

The expression $\frac{n(n-1)(n-2) \ldots (n-r+1)}{\underline{|r}}$ is denoted by ${}^n\mathrm{C}_r$.

Thus ${}^n\mathrm{C}_3 = \frac{n(n-1)(n-2)}{\underline{|3}}$. ${}^n\mathrm{C}_5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}}$.

$${}^n\mathrm{C}_1 = n.$$

With this notation the binomial theorem may be written thus:

$$(a+x)^n = a^n + {}^n\mathrm{C}_1a^{n-1}x + {}^n\mathrm{C}_2a^{n-2}x^2 + {}^n\mathrm{C}_3a^{n-3}x^3 + \ldots.$$

The $(n+1)^{\text{th}}$ term $= {}^n\mathrm{C}_na^{n-n}x^n = \frac{n(n-1)(n-2) \ldots 2 \cdot 1}{\underline{|n}} a^0x^n$

$$= x^n.$$

In the $(n+2)^{\text{th}}$ term the last factor in the numerator of the coefficient would be $1 - 1$, *i.e.* zero.

$\therefore$ the $(n+2)^{\text{th}}$ term and the following terms vanish.

Thus there are $n+1$ terms in the expansion of $(a+x)^n$.

The following is useful.

$${}^n\mathrm{C}_5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}}$$

$$= \frac{n(n-1)(n-2)(n-3)(n-4)\,\underline{|n-5}}{\underline{|5}\,\underline{|n-5}} = \frac{\underline{|n}}{\underline{|5}\,\underline{|n-5}}$$

And generally,

$$^nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r}$$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)\lfloor n-r}{\lfloor r \lfloor n-r} = \frac{\lfloor n}{\lfloor r \lfloor n-r}.$$

Writing $n-r$ for r, we have

$$^nC_{n-r} = \frac{\lfloor n}{\lfloor n-r \lfloor n-n+r} = \frac{\lfloor n}{\lfloor n-r \lfloor r} = {}^nC_r.$$

315. *To find the general term in the expansion of* $(a+x)^n$.

We take the general term to be the $(r+1)^{th}$, as that contains x^r.

We see that the third term $= {}^nC_2 a^{n-2}x^2$.

The fourth ...... $= {}^nC_3 a^{n-3}x^3$.

$\therefore$ the $(r+1)^{th}$ term $= {}^nC_r a^{n-r}x^r$

$$= \frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r} a^{n-r}x^r.$$

If we had to write down the general term of $(a-x)^n$, we should have to write $-x$ for x;

$\therefore$ the general term of $(a-x)^n$

$$\frac{n\,.\,\overline{n-1}\dots\overline{n-r+1}}{\lfloor r} a^{n-r}(-x)^r$$

$$= (-1)^r . \frac{n\,.\,\overline{n-1}\dots\overline{n-r+1}}{\lfloor r} a^{n-r}x^r.$$

Using T_r to denote the r^{th} term in a binomial expansion we notice that T_{r+1} in the expansion of $(a+x)^n$ contains x^r.

Also the expansion is homogeneous with respect to a and x, *i.e.* the sum of their indices is the same in all terms and is equal to n.

Thus the term containing x^r must contain a^{n-r}.

$\therefore$ in the expansion of $(a+x)^n$,

$$T_{r+1} = {}^nC_r a^{n-r}x^r = \frac{n\,.\,\overline{n-1}\dots\overline{n-r+1}}{\lfloor r} a^{n-r}x^r. \quad \text{............(1)}$$

$$\mathsf{T}_r = \frac{n\,.\,\overline{n-1}\,\ldots\,\overline{n-r+2}}{\lfloor r-1}\,a^{n-r+1}x^{r-1};$$

$$\therefore\ \mathsf{T}_{r+1} = \frac{n-r+1}{r}\cdot\frac{x}{a}\cdot\mathsf{T}_r.$$

We can thus derive any term from the preceding term.

This is a general statement of the method employed in Art. 313.

T_{r+1} may be put in the form $\dfrac{\lfloor n}{\lfloor r\,\lfloor n-r}a^{n-r}x^r$ by multiplying numerator and denominator of the coefficient in (1) by $\lfloor n-r$.

Examples. XLIX. a. (*Chiefly oral.*)

In the expansion of $(a+x)^n$:

1. How many terms are there?
2. Which term contains x^5?
3. What power of a accompanies x^5?
4. What is the coefficient of the 3rd term?
5. .... 5th term?
6. Why is the $(r+1)^{th}$ term chosen for the *general* term?
7. What is the denominator of the coefficient of this term?
8. When you have expanded $(a+x)^n$, how can you alter it into $(a-x)^n$?
9. If there are 17 terms, *which* is the middle term?
10. Name in order the coefficients in the expansion of $(1+x)^4$.
11. .... $(1+x)^5$.
12. .... $(1+x)^6$.
13. What multiplier will give the $(r+1)^{th}$ term from the r^{th} in $(1+x)^n$?
14. What multiplier in the case of $(a+x)^n$?
15. Which is the term independent of x in the expansion of $\left(x+\dfrac{1}{x}\right)^6$?
16. What is its value?
17. Find the sum of the coefficients of $(1+x)^8$ by putting $x=1$.
18. $(a+x)^n = a^n + (\text{terms containing a factor } n) + x^n$. If in this we put $n=0$ and deduce that $(a+x)^0 = a^0 + 0 + x^0$, *i.e.* $1 = 1+1$, where is our mistake?
19. In the expansion of $(\sqrt{3}+\sqrt{2})^6$ which terms are rational?
20. In $(a+x)^{12}$ which term has the same coefficient as the 5th?
21. .... 10th?
22. In $(a+x)^n$ one term contains a^3x^7. What is n?

How many factors are there in the product

23. $n(n-1)(n-2)\dots(n-9)$? **24.** $n(n-1)(n-2)\dots(n-p)$?

25. $n(n-1)(n-2)\dots(n-r+1)$? **26.** $n(n-1)(n-2)\dots(n-r+2)$?

27. $n(n-1)(n-2)\dots(n-r-1)$?

28. In the expansion of $(a+x)^n$, what number from the beginning is:

(i) the third term from the end? (ii) the fifth term from the end?

(iii) the r^{th}.................... ..? (iv) the $\overline{r+1}|^{\text{th}}$....................?

29. In the expansion of $(a+x)^{n-1}$, what number from the beginning is:

(i) the fourth term from the end?

(ii) the seventh......................?

(iii) the r^{th} ....?

(iv) the $(r+1)^{\text{th}}$......................?

30. (i) Which is the middle term in the expansion of $(x+a)^{2n}$?

(ii) Write down its value.

(iii) Write down the term immediately preceding the middle term.

(iv) .... following ....

31. Write down the last four terms in the expansion of $(a+x)^n$ in ascending powers of x.

32. (i) Write down the r^{th} term in the expansion of $(x+z)^{25}$.

(ii) ....$\overline{r+1}|^{\text{th}}$.....................

(iii) . ...$\overline{r-1}|^{\text{th}}$.....................

(iv) ... .r^{th} term from the end.

(v) ....$\overline{r+1}|^{\text{th}}$.....................

316. *To prove the Binomial Theorem for a positive integral index.* (Proof by induction.)

Assume it true when the index is n,

i.e. $$(a+x)^n = a^n + {}^nC_1 x a^{n-1} + {}^nC_2 x^2 a^{n-2} + \dots + {}^nC_r x^r a^{n-r} + \dots,$$

where $${}^nC_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{\quad}$$

Multiply both sides by $a+x$;

$$\therefore\ (a+x)^{n+1} = a^{n+1} + {}^nC_1 x a^n + {}^nC_2 x^2 a^{n-1} + \dots + {}^nC_r x^r a^{n-r+1} + \dots$$
$$+ x a^n + {}^nC_1 x^2 a^{n-1} + \dots + {}^nC_{r-1} x^r a^{n-r+1} + \dots$$
$$= a^{n+1} + ({}^nC_1 + 1) x a^n + ({}^nC_2 + {}^nC_1) x^2 a^{n-1} + \dots$$
$$+ ({}^nC_r + {}^nC_{r-1}) x^r a^{n-r+1} + \dots.$$

But ${}^nC_r + {}^nC_{r-1}$

$$= \frac{n.\overline{n-1}.\overline{n-2}\ldots\overline{n-r+1}}{\lfloor r} + \frac{n.\overline{n-1}.\overline{n-2}\ldots\overline{n-r+2}}{\lfloor r-1}$$

$$= \frac{n.\overline{n-1}\ldots\overline{n-r+2}}{\lfloor r-1}\left\{\frac{n-r+1}{r}+1\right\}$$

$$= \frac{n.\overline{n-1}\ldots\overline{n-r+2}}{\lfloor r-1}\cdot\frac{n+1}{r}$$

$$= \frac{\overline{n+1}.n.\overline{n-1}\ldots\overline{n-r+2}}{\lfloor r} = {}^{n+1}C_r.$$

Similarly ${}^nC_1 + 1 = {}^{n+1}C_1$, ${}^nC_2 + {}^nC_1 = {}^{n+1}C_2$, etc.;

$\therefore\ (a+x)^{n+1} = a^{n+1} + {}^{n+1}C_1 xa^n + {}^{n+1}C_2 x^2a^{n-1} + \ldots + {}^{n+1}C_r x^r a^{n-r+1} + \ldots.$

Thus, if the theorem hold for index n, it holds for $n+1$.

But it does hold for index 2; for

$$(a+x)^2 = a^2 + 2ax + \frac{2.1}{1.2}x^2 = a^2 + 2ax + x^2.$$

$\therefore$ the theorem is also true for the index 3;

$\therefore$ .... 4, and so on,

i.e. the theorem is true for all positive integral values of the index.

317. *To prove the Binomial Theorem for a positive integral index.* (2[nd] method, involving a knowledge of Combinations.)

$$(a+x)^n = (a+x)(a+x)(a+x)\ldots \text{ to } n \text{ factors.}$$

The product of these is a homogeneous expression of the n^{th} degree.

The term containing x^r contains also a^{n-r}, and we have to determine its coefficient.

This term is found from the above product by selecting x from r of the n factors, and a from each of the $n-r$ remaining factors.

The x can be selected in ${}^nC_r$ ways, where ${}^nC_r$ denotes the number of combinations of n things taken r at a time.

$\therefore$ the term x^ra^{n-r} appears multiplied by ${}^nC_r$. Similarly for the other terms.

$$\therefore\ (a+x)^n = a^n + {}^nC_1a^{n-1}x + {}^nC_2a^{n-2}x^2 + \ldots + x^n$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{\lfloor 2}a^{n-2}x^2 + \ldots + x^n.$$

318. *To find the greatest term in the expansion of* $(1+x)^n$, *n being a positive integer.*

$$T_{r+1}=\frac{n-r+1}{r}x.T_r=\left(\frac{n+1}{r}-1\right)x.T_r;$$

$\therefore$ T_r is greatest, or equal to the greatest term,

$$\text{when } \left(\frac{n+1}{r}-1\right)x \text{ is first } < \text{ or } = 1,$$

i.e. $\quad .. \quad \dfrac{(n+1)x}{r} \quad .... \quad x+1,$

$$(n+1)x \qquad\qquad r(x+1),$$

$$> \text{ or } = \frac{(n+1)x}{x+1}.$$

(1) Let $\dfrac{(n+1)x}{x+1}$ be an integer, say p.

Then T_{r+1} is greater than T_r as long as $r<p$; (for instance when $r=p-1$ or $p-2$ or any number below p);

i.e. $T_p >$ any term before it.

When $r=p$ the multiplier becomes 1;

$$\therefore\ T_{p+1}=T_p.$$

Thus the terms increase till we get to the p^{th}, the next term is equal to this, and after that they decrease;

$\therefore$ T_p, T_{p+1} are the greatest terms.

(2) Let $\dfrac{(n+1)x}{x+1}$ be a fraction. Let its integral part be q.

Here the multiplier >1 as long as $r<q+$ a fraction,

i.e. the terms increase.

For values of r from $q+1$ onwards the terms decrease.

Thus $\qquad T_{q+1}>$ any term before,

and $\qquad T_{q+1}>T_q$ (for the multiplier >1);

$\therefore$ T_{q+1} is the greatest term.

The greatest term of $(a+x)^n$ may be found by putting it in the form $a^n\left(1+\dfrac{x}{a}\right)^n$. For in this case we have only to find which is the greatest term in $\left(1+\dfrac{x}{a}\right)^n$; and this is done by using $\dfrac{x}{a}$ instead of x in this article.

319. *To find the greatest coefficient in the expansion of* $(a+x)^n$.

This is the same as finding for what value of r ${}^nC_r$ is greatest. See Art. 305.

Example 1. To find the coefficient of x^{17} in the expansion of $(x^2-2x)^{12}$.

$$(x^2-2x)^{12}=x^{12}(x-2)^{12}.$$

$\therefore$ we have to find the coefft. of x^5 in the expansion of $(x-2)^{12}$.

The *term* involving x^5 in this expansion $={}^{12}C_7x^5(-2)^7$. (Art. 315.)

$$\begin{aligned}\therefore \text{ the required coefft.} &= {}^{12}C_7\times(-2)^7\\ &={}^{12}C_5\times(-2)^7\\ &=-\frac{12.11.10.9.8}{1.2.3.4.5}\cdot 2^7\\ &=99\times 2^{10}.\end{aligned}$$

Example 2. In the expansion of $\left(x-\dfrac{3}{x}\right)^{14}$, to find the term which is independent of x.

$$\left(x-\frac{3}{x}\right)^{14}=\frac{1}{x^{14}}(x^2-3)^{14}.$$

$\therefore$ the coefficient of x^{14} in the expansion of $(x^2-3)^{14}$ is the required quantity.

The term involving x^{14} [*i.e.* $(x^2)^7$] in the expansion of $(x^2-3)^{14}$

$$={}^{14}C_7(x^2)^7(-3)^7. \quad \text{(Art. 315.)}$$

$$\therefore \text{ the required term}={}^{14}C_7(-3)^7=-\frac{\lfloor 14}{\lfloor 7\,\lfloor 7}\cdot 3^7.$$

Example 3. To find the greatest term in the expansion of $(2+3x)^9$ when $x=\frac{1}{4}$.

$$(2+3x)^9=2^9\left(1+\frac{3x}{2}\right)^9.$$

$\therefore$ if T_r denotes the r^{th} term,

$$\begin{aligned}T_{r+1}&=\frac{9.8\,..\,(9-r+1)}{\lfloor r}\cdot\left(\frac{3x}{2}\right)^r\cdot 2^9\\ &=\frac{10-r}{r}\cdot\frac{3x}{2}\cdot T_r.\end{aligned}$$

$\therefore$ T_r is greatest, or equal to the greatest,

$$\text{when } \frac{10-r}{r}\cdot\frac{3x}{2} \text{ is first} < \text{or}=1,$$

$$\textit{i.e.} \text{ when } \frac{10-r}{r}\cdot\frac{3}{8} \;\ldots\ldots\ldots < \text{or}=1,$$

$$\ldots\ldots\ldots\ldots \frac{10}{r}-1 \;\ldots\ldots\ldots < \text{or}=\frac{8}{3},$$

$$\ldots\ldots\ldots\ldots \frac{10}{r} \;\ldots\ldots\ldots < \text{or}=\frac{11}{3},$$

i.e. when $\frac{1}{r}$ is first $<$ or $=\frac{11}{30}$,

.... r .... $>$ or $=\frac{30}{11}$,

i.e. when $r=3$.

$\therefore$ the 3rd term is the greatest,

and its value is $2^9\frac{9.8}{1.2}\cdot\left(\frac{3x}{2}\right)^2$,

which $=2^9\frac{9.8}{1.2}\cdot\frac{3^2}{8^2}=\frac{2^9.81}{2^4}=2592$.

Example 4. Find approximate values of $(0{\cdot}999)^7$.

$$(0{\cdot}999)^7=\left(\frac{999}{1000}\right)^7=\left(1-\frac{1}{10^3}\right)^7$$

$$=1-\frac{7}{10^3}+\frac{7.6}{1.2}\cdot\frac{1}{10^6}-\frac{7.6.5}{1.2.3}\cdot\frac{1}{10^9}+\ldots$$

$$=1-{\cdot}007+{\cdot}000021-{\cdot}000000035+\ldots$$

$=1-{\cdot}007$, *i.e.* ${\cdot}9930$ correct to four decimal places,

or $={\cdot}9930210$ correct to seven decimal places.

Example 5. Find the coefficient of x^7 in the expansion of $(1-x)^4(1+x)^9$.

The coefft. of x^7 in $(1-x)^4(1+x)^9$

$=$ .... $(1-4x+6x^2-4x^3+x^4)(1+x)^9$

$=$ the coefft. of x^7 in $(1+x)^9$

$-4\times$ the coefft. of x^6 in $(1+x)^9$

$+6\times$ .... x^5 ....

$-4\times$ .... x^4 ....

$+$ .... x^3 ....

$$=\frac{9.8}{1.2}-4\times\frac{9.8.7}{1.2.3}+6\times\frac{9.8.7.6}{1.2.3.4}-4\times\frac{9.8.7.6}{1.2.3.4}+\frac{9.8.7}{1.2.3}$$

$$=36-2.3.8.7+\frac{2.9.8.7.6}{1.2.3.4}+84$$

$$=120-336+252=372-336=36.$$

Examples. XLIX. b.

Expand

1. $(1+x)^5$.
2. $(a+x)^7$.
3. $(a^2-x^2)^6$.
4. $\left(ab-\frac{x}{2}\right)^8$.
5. $\left(x+\frac{1}{x}\right)^8$.
6. $\left(ax-\frac{b}{x}\right)^7$.
7. $(3x+2y)^5$.
8. $(a+b\sqrt{2})^8+(a-b\sqrt{2})^8$.

9. Give the 5th term in the expansion of $(1-x)^{10}$.

10. .... 6th .... $(a^2+x^2)^{11}$.

11. .... 12th .... $(3-y)^{15}$.

12. Give the middle term in the expansion of $\left(x+\frac{1}{x}\right)^{12}$.

13. Give the two middle terms in the expansion of $\left(2x-\frac{1}{x}\right)^{11}$.

14. n being a positive integer, find T_{r+1} in the expansion of $(1+x)^{2n}$ and T_{r+1} in the expansion of $(3-2x)^n$.

15. A = the sum of the odd terms, B the sum of the even terms in the expansion of $(a+x)^n$; prove that $A^2-B^2=(a^2-x^2)^n$.

16. The coefficient of x^n in $(1+x)^{2n}=\frac{\lfloor 2n}{\lfloor n \lfloor n}=\frac{1.3.5\ldots(2n-1)}{\lfloor n}2^n$.

17. In the expansion of $(1+x)^n$ given x^2. $T_{r+1}=T_{r+3}$, find r.

Write down the general term in

18. $(1+2x)^n$.

19. $(3-5x)^{2n}$.

20. $(a^2-3bx)^n$.

21. $\left(x+\frac{1}{x}\right)^n$.

22. Write down the middle term of $\left(x^2-\frac{1}{x^2}\right)^{2n}$.

23. Write down the two middle terms of $(2x+y)^9$.

24. Find the coefficient of x^7 in $(2x+x^2)^5$.

25. Find the coefficient of x^8 in $\left(x^2+\frac{1}{x}\right)^7$.

26. Find the term independent of x in $\left(x+\frac{2}{x}\right)^{10}$.

27. Show that $(a+\sqrt{2})^n+(a-\sqrt{2})^n$ contains no surds.

28. Expand $(a-x)^7(a+x)^2$.

29. Expand $\left(x-\frac{1}{x}\right)^5\left(x+\frac{1}{x}\right)^3$.

30. Simplify $(\sqrt{3}+1)^8-(\sqrt{3}-1)^8$.

31. Prove that $\left(x+\frac{1}{x}\right)^6-\left(x-\frac{1}{x}\right)^6=12\left(x^4+\frac{1}{x^4}+\frac{10}{3}\right)$.

32. Show by actual multiplication that the whole coefficient of x^4 in the product of the expansions of $(1+x)^n$ and $(1-x)^n$ = the coefficient of x^4 in the expansion of $(1-x^2)^n$.

33. Find the ratio of the middle term to the one before it in the expansion of $(1-2x)^{20}$.

34. If a_r denote the coefficient of x^r in $(1-x)^{2n-1}$, prove that

$$a_{r-1}+a_{2n-r}=0.$$

35. Expand $(1+a_1)(1+a_2)(1+a_3)(1+a_4)$, and show how $(1-a)^4$ may be derived from the result.

36. Find the greatest coefficient in the expansion of $(1+x)^{14}$.

37. Find the greatest term in the expansion of $(\frac{1}{2}+x)^8$ when $x=1$.

38. .... $(3+2b)^9$ when $b=2$.

39. .... $(4+9x)^6$ when $x=1$.

40. .... $(1+x)^6$ when $x=\frac{1}{2}$.

41. .... $(\frac{1}{2}+\frac{1}{3}x)^7$ when $x=1$.

Find the values of the following:

42. $(1 \cdot 05)^2$ correct to two decimal places.

43. $(1 \cdot 05)^3$ ....

44. $(1 \cdot 02)^3$ ....

45. $(1 \cdot 002)^3$ correct to three decimal places.

46. $(1 \cdot 04)^4$ ....

47. $(1 \cdot 002)^6$ .... five ....

48. $(\cdot 999)^3$ ....

49. $(\cdot 98)^4$ .... four ....

50. $(\cdot 9997)^4$ ....

51. $(1 \cdot 0005)^7$ .... six ....

52. Prove that $\left(x+\frac{1}{x}\right)^{20}=x^{20}+\frac{1}{x^{20}}+{}^{20}C_1\left(x^{18}+\frac{1}{x^{18}}\right)+{}^{20}C_2\left(x^{16}+\frac{1}{x^{16}}\right)+\ldots,$ and give the last term.

53. The coefficients of the 5th and 7th terms of $(1+x)^n$ are 70 and 28; find n.

54. Find the coefficient of x^9 in the expansion of $(1-2x)^5(1+x)^8$.

55. .... x^5 .... $(1+x)^3(1-x)^6$.

56. .... x^5 .... $(1+x+x^2)(1-2x)^7$.

57. .... x^3 .... $(1+2x)^3(1+3x)^4$.

320. *To find the sum of the coefficients in the expansion of* $(1+x)^n$.

Write $(1+x)^n$ in the form $c_0+c_1x+c_2x^2+c_3x^3+\ldots+c_nx^n$.

Put $x=1$. Then $c_0+c_1+c_2+\ldots+c_n=(1+1)^n=2^n$.

To prove that the sum of the even coefficients is equal to the sum of the odd coefficients in the expansion of $(1+x)^n$.

Writing $(1+x)^n$ in the form

$$c_0+c_1x+c_2x^2+\ldots+c_nx^n,$$

and putting $x=-1$, we have

$$c_0-c_1+c_2-c_3+\ldots+(-1)^nc_n=(1-1)^n=0;$$

i.e. the sum of the even coeffts. = the sum of the odd coeffts.

$$=\frac{2^n}{2}\text{ (proved above)}=2^{n-1}.$$

321. *To find the value of* $c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2$.

$$(1+x)^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n,$$

$$(x+1)^n = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + \dots + c_n;$$

$\therefore$ by multiplication, $c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2 =$ coefft. of x^n in the product $(1+x)^n(x+1)^n$, *i.e.* in the expansion of $(1+x)^{2n}$;

$$\therefore\ c_0^2 + c_1^2 + c_2^2 + \dots + c_n^2 = \frac{\lfloor 2n}{\lfloor n \lfloor n}.$$

Examples. XLIX. c.

1. Find the sum of the coefficients in the expansion of $(2x+3y)^n$.

2. .... $(5a-4b)^n$.

When n is a positive integer let the expansion of $(1+x)^n$ be written as

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n,$$

and prove the following:

3. $c_1 + c_3 + c_5 + \dots = 2^{n-1}$.

4. $c_1 + 2c_2 + 3c_3 + \dots + nc_n = n \cdot 2^{n-1}$.

5. $c_0 + \frac{1}{2}c_1 + \frac{1}{3}c_2 + \dots \frac{c_n}{n+1} = \frac{2^{n+1}-1}{n+1}$.

6. $c_0c_1 + c_1c_2 + c_2c_3 + \dots + c_{n-1}c_n = \frac{\lfloor 2n}{\lfloor n-1 \lfloor n+1}$.

[Coefficient of x^{n+1} in $(1+x)^n(x+1)^n$.]

7. $c_0c_r + c_1c_{r+1} + c_2c_{r+2} + \dots + c_{n-r}c_n = \frac{\lfloor 2n}{\lfloor n-r \lfloor n+r}$.

8. $c_0 - \frac{1}{2}c_1 + \frac{1}{3}c_2 - \dots + (-1)^n \frac{c_n}{n+1} = \frac{1}{n+1}$.

9. The sum of the products of c_0, c_1, c_2, etc. ... taken two at a time

$$= 2^{2n-1} - \frac{\lfloor 2n}{2\lfloor n \lfloor n}.$$

10. Sum the series $1 + {}^nC_2 x^2 + {}^nC_4 x^4 + \dots$.

11. Simplify $1 + {}^nC_1 \cdot \frac{x}{1+x} + {}^nC_2 \left(\frac{x}{1+x}\right)^2 + \dots + \left(\frac{x}{1+x}\right)^n$.

CONVERGENT AND DIVERGENT SERIES.

322. When a series of quantities has a limited number of terms, it is called a **finite series**; when the number of terms is unlimited, it is called an **infinite series.**

When the sum of the first n terms of a series cannot exceed a finite limit S, however great n may be, the series is said to be **Convergent.**

If the sum continually approaches and ultimately becomes indefinitely near to S as n increases, S is said to be the sum to infinity.

When the sum of the first n terms increases without limit as n increases indefinitely, the series is said to be **Divergent.**

We have seen (Art. 274) that if x is numerically less than unity, the sum of n terms of the series $1+x+x^2+x^3+\ldots$ cannot exceed $\frac{1}{1-x}$, and, as n increases, the sum continually approaches and ultimately becomes indefinitely near to $\frac{1}{1-x}$; therefore this series is convergent, and $\frac{1}{1-x}$ is its sum to infinity.

If $x=1$, the series becomes $1+1+1+1+\ldots$; and n, the sum of n terms, increases without limit as n increases. Therefore this series is divergent.

323. *An infinite series, whose terms are alternately positive and negative, is convergent if each term is numerically less than the preceding term.*

Let $u_1-u_2+u_3-u_4+\ldots$ be the series, and let S denote the sum of any number of terms.

The series may be written

$$(u_1-u_2)+(u_3-u_4)+(u_5-u_6)+\ldots,$$

which shows that S is a positive quantity, for all the expressions in the various brackets are positive.

It also may be written

$$u_1-(u_2-u_3)-(u_4-u_5)-(u_6-u_7)-\ldots,$$

which shows that S is always less than u_1, however great the number of terms may be.

$\therefore$ the series is convergent.

324. *An infinite series is convergent if, from and after any fixed term, the ratio of each term to the preceding term is numerically less than some quantity which is itself numerically less than unity.*

Let $u_1+u_2+u_3+u_4+\ldots$ be the series, where u_1 is the fixed term, and let S denote the sum of these terms.

Then
$$S=u_1+u_2+u_3+u_4+\ldots$$
$$=u_1\left[1+\frac{u_2}{u_1}+\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\frac{u_4}{u_3}\cdot\frac{u_3}{u_2}\cdot\frac{u_2}{u_1}+\ldots\right].$$

By hypothesis, each of the ratios $\frac{u_2}{u_1}, \frac{u_3}{u_2}, \frac{u_4}{u_3}\ldots$ is less than k, where k is a quantity numerically less than unity.

$$\therefore\ S<u_1(1+k+k^2+k^3+\ldots),$$

that is $<\frac{u_1}{1-k}$. (Art. 274.)

$\therefore$ the series is convergent.

325. *To prove that the series*

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\tfrac{1}{5}+\ldots \textit{ is divergent.}$$

The first term $=1$.

.. second ...... $=\frac{1}{2}$.

... next 2 terms are together $>\frac{2}{4}$.

... . 4 .... $>\frac{4}{8}$ [for the least is $\frac{1}{8}$]

.... 8 . ... $>\frac{8}{16}$, and so on.

$\therefore$ the series $>1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$ to infinity. Hence it is divergent.

Examples. XLIX. d.

1. Prove that the series
$$1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\tfrac{1}{5}\ldots \text{ is convergent.}$$

Prove that the following series are convergent.

2. $\frac{1}{2}-\frac{1}{2}\cdot\frac{1}{2^2}+\frac{1}{3.2^3}-\frac{1}{4.2^4}+\ldots.$ **3.** $1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\frac{1}{\lfloor 5}+\ldots.$

4. $1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\ldots$ for finite values of x.

5. $1+2x+3x^2+4x^3+5x^4+\ldots$ if x is numerically less than 1.

6. Use the method of Art. 325 to prove that the series
$$\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\frac{1}{4^p}+\ldots \text{ is convergent if } p>1.$$

326. *The Binomial Theorem when the exponent (or index) is not a positive integer.*

The Theorem has been proved to be true if the index be any positive integer. It remains to examine its truth for *any* value of n, where the expression $(1+x)^n$ has to be expanded.

When n is a positive integer the expansion terminates; for the coefficients contain factors n, $n-1$, $n-2$, etc., and we reach a coefficient which vanishes, and thus the end of the series is reached.

When n is negative or fractional the series of terms goes on for ever, and it will be found that the expansion is not always convergent unless x is numerically less than unity.

Under certain conditions the series is convergent when $x=1$ or -1, but it is not necessary to consider those cases at this stage.

To prove that the Binomial Series is convergent for all negative and fractional values of the index, provided that x is numerically less than unity.

In the expansion of $(1+x)^n$

$$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\cdot x=\left(\frac{n+1}{r}-1\right)x.$$

As r increases, $\frac{n+1}{r}$ continually decreases *numerically* whatever value n may have.

$\therefore$ as r increases, $\frac{n+1}{r}-1$ continually approaches the value -1.

$\therefore$ as r increases indefinitely, the ratio $\left(\frac{n+1}{r}-1\right)x$ continually approaches $-x$, and eventually will be numerically less than some quantity which is itself numerically less than unity.

$\therefore$ the series is convergent. (Art. 324.)

327. We shall use $f(n)$ to denote the series

$$1+nx+\frac{n(n-1)}{\underline{|2}}x^2+\frac{n(n-1)(n-2)}{\underline{|3}}x^3+\ldots$$

for any value of n. Before reading the next few articles the

student would do well to write down the series for several particular values of n; say,

$$f(7),\quad f(8),\quad f(-2),\quad f(-3),\quad f(\tfrac{1}{2}),\quad f(m+n).$$

We have proved that $f(n)=(1+x)^n$ if n is a positive integer, and we shall now prove that

$$f\left(\frac{r}{s}\right)=(1+x)^{\frac{r}{s}},\text{ where } r \text{ and } s \text{ are positive integers,}$$

and that $f(-n)=(1+x)^{-n}$ ... n is positive (integral or fractional).

328. *If a and b are positive integers, and $a+b>r$, then*

$$^{a+b}C_r={}^aC_r+{}^aC_{r-1}\,{}^bC_1+{}^aC_{r-2}\,{}^bC_2+\ldots+{}^bC_r,$$

where ^pC_r denotes the number of combinations of p things taken r at a time.

a and b are positive integers; therefore by the Binomial Theorem,

$$(1+x)^a\equiv 1+{}^aC_1x+{}^aC_2x^2+\ldots+{}^aC_rx^r+\ldots+x^a,$$

and $$(1+x)^b\equiv 1+{}^bC_1x+{}^bC_2x^2+\ldots+{}^bC_rx^r+\ldots+x^b.$$

$\therefore$ by multiplication,

$(1+x)^{a+b}\equiv$ the product of the two above finite series.

$\therefore$ the coeffts. of x^r on the two sides are equal,

i.e. $^{a+b}C_r={}^aC_r+{}^aC_{r-1}\,{}^bC_1+{}^aC_{r-2}\,{}^bC_2+\ldots+{}^bC_r.$ Q.E.D.

VANDERMONDE'S THEOREM.

329. Let a_r denote the product $a(a-1)(a-2)\ldots(a-r+1)$, where a *has any value*, integral, fractional, positive or negative.

If a is a positive integer, $\dfrac{a_r}{\lfloor r}=\dfrac{a(a-1)(a-2)\ldots(a-r+1)}{\lfloor r}={}^aC_r,$

i.e. the number of r-combinations of a things.

We will now proceed to prove Vandermonde's theorem, which states that:

If n be any positive integer, and a, b have any values whatever,

then $$(a+b)_n=a_n+na_{n-1}b_1+\frac{n(n-1)}{\lfloor 2}\cdot a_{n-2}b_2+\ldots+b_n.$$

We have proved in the preceding article, that if a and b are positive integers, and $a+b>n$,

$$^{a+b}C_n = {}^aC_n + {}^aC_{n-1}{}^bC_1 + {}^aC_{n-2}{}^bC_2 + \dots + {}^bC_n;$$

i.e. with the above notation,

$$\frac{(a+b)_n}{\lfloor n} = \frac{a_n}{\lfloor n} + \frac{a_{n-1}}{\lfloor n-1}\frac{b_1}{\lfloor 1} + \frac{a_{n-2}}{\lfloor n-2}\cdot\frac{b_2}{\lfloor 2} + \dots + \frac{b_n}{\lfloor n}.$$

∴ multiplying both sides by $\lfloor n$, we have

$$(a+b)_n = a_n + na_{n-1}b_1 + \frac{n(n-1)}{\lfloor 2}\cdot a_{n-2}b_2 + \dots + b_n.$$

Both sides of this equation are of the n^{th} degree in a and b, and we know that it is true for all positive integral values of a and b. We have to prove it true for all values of a and b.

If a is *any particular* integer greater than n, the two sides of this equation are equal for *any* positive integral value of b, *i.e.* the equation is true for *more than n values of b*. ∴ it is true for that value of a and for *any value of b*. (Art. 208.)

In the same way, the equation is true for any particular value of b and for more than n values of a.

∴ it is true for that value of b and for *any value of a*.

∴ it is true for all values of a and for all values of b.

330. *Proof of the Binomial Theorem for any index.*

Using the notation of Art. 329,

let $f(m)$ denote the series $1 + \frac{m_1x}{\lfloor 1} + \frac{m_2x^2}{\lfloor 2} + \frac{m_3x^3}{\lfloor 3} + \dots,$

so that $$f(m) \equiv 1 + \frac{m_1x}{\lfloor 1} + \frac{m_2x^2}{\lfloor 2} + \frac{m_3x^3}{\lfloor 3} + \dots + \frac{m_rx^r}{\lfloor r} + \dots,$$

$$f(n) \equiv 1 + \frac{n_1x}{\lfloor 1} + \frac{n_2x^2}{\lfloor 2} + \frac{n_3x^3}{\lfloor 3} + \dots + \frac{n_rx^r}{\lfloor r} + \dots,$$

and $$f(m+n) \equiv 1 + \frac{(m+n)_1x}{\lfloor 1} + \frac{(m+n)_2x^2}{\lfloor 2} + \dots + \frac{(m+n)_rx^r}{\lfloor r} + \dots.$$

In the product $f(m)\times f(n)$, the coefficient of x^r

$$=\frac{m_r}{\lfloor r}+\frac{m_{r-1}}{\lfloor r-1}\cdot\frac{n_1}{\lfloor 1}+\frac{m_{r-2}}{\lfloor r-2}\cdot\frac{n_2}{\lfloor 2}+\ldots+\frac{n_r}{\lfloor r}$$

$$=\frac{1}{\lfloor r}\left[m_r+rm_{r-1}n_1+\frac{r(r-1)}{\lfloor 2}m_{r-2}n_2+\ldots+n_r\right]$$

$$=\frac{1}{\lfloor r}\cdot(m+n)_r \text{ by Vandermonde's Theorem}$$

$$=\text{the coefficient of } x^r \text{ in } f(m+n),$$

i.e. the coefficient of *any* power of x in $f(m)\times f(n)$

= .... the same .... $f(m+n)$.

$\therefore$ *$f(m)\times f(n)=f(m+n)$ for all values of m and n*, for all three series are convergent if x is numerically less than unity.

Also, on the same supposition,

$$f(m)\times f(n)\times f(p)=f(m+n)\times f(p)=f(m+n+p), \text{ and so on.}$$

$$\therefore\ f(m)\times f(n)\times f(p)\times\ldots=f(m+n+p+\ldots).$$

In this equation let $m=n=p=\ldots=\frac{r}{s}$, where r and s are positive integers, and let there be s of these quantities.

$$\therefore\ f\left(\frac{r}{s}\right)\times f\left(\frac{r}{s}\right)\times\ldots \text{ to } s \text{ factors}=f\left(\frac{r}{s}+\frac{r}{s}+\frac{r}{s}+\ldots \text{ to } s \text{ terms}\right)$$

$$\therefore\ \left[f\left(\frac{r}{s}\right)\right]^s=f(r)=(1+x)^r, \text{ for } r \text{ is a positive integer.}$$

$$\therefore\ (1+x)^{\frac{r}{s}}=f\left(\frac{r}{s}\right)=1+\frac{r}{s}x+\frac{\frac{r}{s}\left(\frac{r}{s}-1\right)}{\lfloor 2}x^2+\ldots,$$

which is the Binomial Theorem with a fractional index.

To prove the theorem for a negative index, put $-m$ for n in the product $f(m)\times f(n)$.

Then $\qquad f(m)\times f(-m)=f(m-m)=f(0)=1.$

$$\therefore\ f(-m)=\frac{1}{f(m)}=\frac{1}{(1+x)^m}=(1+x)^{-m}$$

$$\therefore\ (1+x)^{-m}=f(-m)=1+(-m)x+\frac{(-m)(-m-1)}{\lfloor 2}x^2+\ldots.$$

331. *Euler's proof of the Binomial Theorem for any index.*

Let $f(m)=1+mx+\frac{m.\overline{m-1}}{\lfloor 2}x^2+\frac{m.\overline{m-1}.\overline{m-2}}{\lfloor 3}x^3+\dots$.

Then $f(n)=1+nx+\frac{n.\overline{n-1}}{\lfloor 2}x^2+\frac{n.\overline{n-1}.\overline{n-2}}{\lfloor 3}x^3+\dots$.

By actual multiplication of these two series we can prove that certainly as far as a few terms

$$f(m)\times f(n)=1+(m+n)x+\frac{\overline{m+n}.\overline{m+n-1}}{\lfloor 2}x^2+\dots.$$

If x were greater than 1, the multiplication of these two series which have an unlimited number of terms, would become unintelligible; for they contain terms which would become infinite. If however x is numerically less than 1, the series are convergent, and we may assume that, as it is mere multiplication, the *form* of the result is the same whatever letters are employed in it, and whatever be the values of those letters.

$\therefore$ to aid our unfinished multiplication we may perform it with m and n treated as positive integers, and say that the result will hold when they are changed to fractional or negative quantities.

Now if m and n were positive integers $f(m)$ and $f(n)$ would be $(1+x)^m$ and $(1+x)^n$.

$\therefore$ $f(m)\times f(n)$ would be $(1+x)^m\times(1+x)^n$, *i.e.* $(1+x)^{m+n}$, *i.e.* $f(m+n)$.

$$\therefore \text{ whatever } m \text{ and } n \text{ may be, } f(m)\times f(n)=f(m+n). \quad\dots(1)$$

The rest of the proof is the same as that by Vandermonde's Theorem in Art. 330.

In both proofs given in this chapter, it is assumed that the product of two convergent series is also a convergent series.

332. The Binomial Theorem can be proved for any *negative integral index* as follows, provided that x is numerically less than unity.

$$\{1+2x+3x^2+4x^3+\dots \text{ to infinity}\}(1-x)$$
$$=1+2x+3x^2+4x^3+5x^4+\dots$$
$$\qquad -x-2x^2-3x^3-4x^4-\dots$$
$$=1+x+x^2+x^3+x^4+\dots \text{ (an infinite G.P.)}$$
$$=\frac{1}{1-x}\ [\text{for } x<1]$$
$$=(1-x)^{-1}.$$
$$\therefore\ 1+2x+3x^2+4x^3+\dots=(1-x)^{-2}.$$

Substituting $-x$ for x, we get

$$1-2x+3x^2-4x^3+\dots \text{ to infinity}=(1+x)^{-2},$$
$$i.e.\ f(-2)=(1+x)^{-2}.$$

Again,

$$\left\{1+(n+1)x+\frac{(n+1)(n+2)}{\lfloor 2}x^2+\frac{(n+1)(n+2)(n+3)}{\lfloor 3}x^3+\dots\right\}(1-x)$$
$$=1+(\overline{n+1}-1)x+\left(\frac{\overline{n+1}\,.\,\overline{n+2}}{\lfloor 2}-\overline{n+1}\right)x^2$$
$$+\left(\frac{\overline{n+1}\,.\,\overline{n+2}\,.\,\overline{n+3}}{\lfloor 3}-\frac{\overline{n+1}\,.\,\overline{n+2}}{\lfloor 2}\right)x^3+\dots$$
$$=1+nx+\frac{n\,.\,\overline{n+1}}{\lfloor 2}x^2+\frac{n\,.\,\overline{n+1}\,.\,\overline{n+2}}{\lfloor 3}x^3+\dots.$$

Putting $-x$ for x, we get

$$\left\{1-\overline{n+1}\,.\,x+\frac{\overline{n+1}\,.\,\overline{n+2}}{\lfloor 2}x^2-\frac{\overline{n+1}\,.\,\overline{n+2}\,.\,\overline{n+3}}{\lfloor 3}x^3+\dots\right\}(1+x)$$
$$=1-nx+\frac{n\,.\,\overline{n+1}}{\lfloor 2}x^2-\frac{n\,.\,\overline{n+1}\,.\,\overline{n+2}}{\lfloor 3}x^3+\dots,$$
$$i.e.\quad f(-\overline{n+1})=f(-n)\,.\,(1+x)^{-1};$$

$\therefore$ if $f(-n)=(1+x)^{-n}$, we see that $f(-\overline{n+1})=(1+x)^{-\overline{n+1}}$.

But we know that $f(-1)=(1+x)^{-1}$ *if* $x<1$,

$$\text{and } f(-2)=(1+x)^{-2};$$
$$\therefore\ f(-3)=(1+x)^{-3} \text{ by induction.}$$

Hence $f(-4)=(1+x)^{-4}$, and so on;

$\therefore$ for all integral values of n, if $x<1$, we have

$$(1+x)^{-n}=f(-n)=1-nx+\frac{n\,.\,\overline{n+1}}{\lfloor 2}x^2-\frac{n\,.\,\overline{n+1}\,.\,\overline{n+2}}{\lfloor 3}x^3+\dots.$$

Example 1. Expand $(1+x)^{-\frac{2}{3}}$ and find the $(r+1)^{\text{th}}$ term.

$$(1+x)^{-\frac{2}{3}}=1+(-\tfrac{2}{3})x+\frac{(-\frac{2}{3})(-\frac{5}{3})}{1\,.\,2}x^2+\frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})}{1\,.\,2\,.\,3}x^3+\dots.$$

$$T_{r+1}=\frac{(-\frac{2}{3})(-\frac{5}{3})(-\frac{8}{3})\dots(-\frac{2}{3}-r+1)}{1\,.\,2\,.\,3\dots r}x^r$$

$$=(-1)^r\cdot\frac{2\,.\,5\,.\,8\dots(3r-1)}{3^r\,.\,\lfloor r}x^r.$$

Example 2. Expand $(1+x)^{-4}$ and find the $(r+1)^{\text{th}}$ term.

$$(1+x)^{-4}=1+(-4)x+\frac{(-4)(-5)}{1\,.\,2}x^2+\frac{(-4)(-5)(-6)}{1\,.\,2\,.\,3}x^3+\dots.$$

$$T_{r+1}=\frac{(-4)(-5)(-6)\dots(-4-r+1)}{1\,.\,2\,.\,3\dots r}x^r$$

$$=(-1)^r\cdot\frac{4\,.\,5\,.\,6\dots r\,.\,(r+1)(r+2)(r+3)}{1\,.\,2\,.\,3\,.\,4\,.\,5\dots r}x^r$$

$$=(-1)^r\cdot\frac{(r+1)(r+2)(r+3)}{1\,.\,2\,.\,3}x^r.$$

Notice that $(1-\mathbf{x})^{-n}$ has all its terms positive.

Also that $(1-\mathbf{x})^{-1}=1+\mathbf{x}+\mathbf{x}^2+\mathbf{x}^3+\dots.$

$$(1-\mathbf{x})^{-2}=1+2\mathbf{x}+3\mathbf{x}^2+\dots+(\mathbf{r}+1)\mathbf{x}^r+\dots.$$

$$(1-\mathbf{x})^{-3}=1+3\mathbf{x}+\frac{3\,.\,4}{1\,.\,2}\mathbf{x}^2+\frac{4\,.\,5}{1\,.\,2}\mathbf{x}^3+\dots$$

$$+\frac{(\mathbf{r}+1)(\mathbf{r}+2)}{1\,.\,2}\mathbf{x}^r+\dots.$$

333. *To find the greatest term without regard to sign in the expansion of $(a\pm x)^{\pm n}$ in ascending powers of x.*

The $(r+1)^{\text{th}}$ term is obtained from the r^{th} (without regard to sign) by the multiplier

$$\frac{n-r+1}{r}\cdot\frac{x}{a}\text{ (in the case of a positive index),}$$

or $$\frac{n+r-1}{r}\cdot\frac{x}{a}\text{ (in the case of a negative index).}$$

$\therefore T_{r+1}>T_r$ as long as this multiplier >1.

Now since the multiplier may be put in the form

$$\left(\frac{n+1}{r}-1\right)\frac{x}{a} \quad \text{or} \quad \left(\frac{n-1}{r}+1\right)\frac{x}{a},$$

it continually decreases as r increases.

The r^{th} term is greatest or equal to the greatest when the multiplier first $<$ or $= 1$,

$$\text{i.e.} \quad (1)\ \frac{n+1}{r}x < \text{or} = a+x, \quad \text{or} \quad (2)\ \frac{n-1}{r}x < \text{or} = a-x.$$

Thus the r^{th} term is the greatest or = the greatest

$$\text{when } r \text{ is first} > \text{or} = \frac{(n+1)x}{a+x} \text{ (with the positive index)},$$

$$\text{or } \ldots\ldots > \text{or} = \frac{(n-1)x}{a-x} \text{ (with the negative index)}.$$

Example 1. Find the greatest term in the expansion of $(1+x)^{\frac{11}{2}}$ where $x=\frac{2}{3}$.

$$T_{r+1}=T_r \cdot \frac{n-r+1}{r} \cdot x = T_r \cdot \frac{\frac{11}{2}-r+1}{r} \cdot \frac{2}{3} = T_r \cdot \frac{13-2r}{3r}.$$

$\therefore$ T_r is greatest, or equal to the greatest term,

$$\text{when } \frac{13-2r}{3r} \text{ is first} < \text{or} = 1,$$

$$\text{i.e. } \ldots\ldots\ldots\ 13-2r \ \ldots\ldots\ldots\ldots = 3r,$$

$$\text{i.e. } \ldots\ldots\ldots\ 5r \ \ldots\ldots\ldots > \text{or} = 13.$$

The smallest integer greater than $\frac{13}{5}$ is 3.

$\therefore$ the *third* term is the greatest.

$$\text{This term} = \frac{\frac{11}{2}\cdot\frac{9}{2}}{\underline{|2}}\left(\frac{2}{3}\right)^2 = \frac{11}{2}.$$

Example 2. Which is the greatest term in the expansion of $(1-x)^{-5}$ where $x=\frac{3}{4}$?

Here T_{r+1} is numerically equal to $T_r \cdot \frac{5+r-1}{r} \cdot \frac{3}{4}$

$$= T_r \cdot \frac{4+r}{r} \cdot \frac{3}{4}$$

$$= T_r \cdot \frac{12+3r}{4r}.$$

$\therefore$ T_r is greatest, or equal to the greatest term,

when $\dfrac{12+3r}{4r}$ is first $<$ or $=1$,

i.e. .... $12+3r$ .... $=4r$,

i.e. .... r .... $>$ or $=12$.

If $r=12$, the multiplier $=1$;

$\therefore$ the 12^{th} and 13^{th} terms are equal and are the greatest.

Example 3. Find the greatest coefficient in the expansion of $(4x+5y)^n$.

By putting $x=y=1$ we reduce the terms to their coefficients.

$\therefore$ it is only necessary to find the greatest *term* in the expansion of $(4+5)^n$.

Examples. XLIX. e.

Expand to 4 terms

1. $(a-2x)^{-5}$.
2. $(1+2x)^{\frac{1}{3}}$.
3. $(1-x)^{-\frac{1}{2}}$.
4. $(1+\frac{2}{3}x)^{-6}$.
5. $(a^2-x^2)^{\frac{1}{2}}$.
6. $(1+\frac{1}{2}x)^{\frac{1}{2}}$.
7. $(1+x^2)^{-\frac{1}{3}}$.
8. $(1-3x)^{-\frac{1}{3}}$.
9. $\dfrac{1}{a^2+b^2}$.

10. Find the 4th term of $(1+x)^{\frac{1}{2}}$.

11. .... 5th .... $(3a-2b)^{-10}$.

12. Find the coefficient of x^{12} in $(1-b^3x^2)^{\frac{5}{2}}$.

13. Find the $(r+1)^{th}$ term of $(x+a)^{-n}$.

14. .... $(1-x)^{-3}$.

15. .. . . $(1-x)^{-5}$.

16. .... $(2-x)^{-5}$.

17. .... $(1+x)^{-\frac{2}{3}}$.

18. The coefficient of x^r in $(1-4x)^{-\frac{1}{2}}$ is $\dfrac{\lfloor 2r}{(\lfloor r)^2}$.

19. Expand to 4 terms $\left(1-\dfrac{1}{x}\right)^{-n}$. Write down the $(n+1)^{th}$ term and find its value when $x=\dfrac{1}{n}$.

20. Find the coefficient of x^r in $\left(x+\dfrac{1}{x}\right)^n$.

21. Find the coefficient of x^{10} in the expansion of $(x+x^2)^{-3}$.

22. .... x^r .... $\dfrac{(3+2x)}{(1-x)^2}$.

23. .... x^r .... $\dfrac{(1-3x)^2}{(1-x)^3}$.

24. .... x^5 .... $\dfrac{1-2x+3x^2}{(1-x)^4}$.

25. Find the coefficient of x^{2r} in the expansion of $\dfrac{1-x}{(1+x)^2}$.

26. .... x^r .... $\dfrac{3-x}{(1-x)^3}$.

27. The coefficient of x^n in the expansion of $(1+x+x^2+\ldots+x^{n-1})^n$

$$= -n + \text{coefficient of } x^n \text{ in } (1+x)^{2n-1}.$$

28. Find the coefficient of x^r in $(1+2x+3x^2+\ldots\ ad\ inf.)^{-n}$.

29. Find the first negative term of $(1+x)^{\frac{3}{2}}$.

30. How many terms of $(1-3x)^{\frac{7}{2}}$ are positive?

31. Which is the greatest term in the expansion of $(1+x)^{\frac{5}{2}}$ when $x=\frac{2}{3}$.

32. .... $(1-x)^{-8}$ when $x=\frac{1}{2}$.

33. .... $(1+2x)^{\frac{7}{2}}$ when $x=\frac{2}{5}$.

34. .. . . $(1+\frac{3}{4})^{\frac{13}{3}}$.

35. Express $\dfrac{1}{5x+1}$ in a series of ascending and also of descending powers of x.

36. Express $\dfrac{1}{(3-5x)^2}$ in a series of ascending and also of descending powers of x.

334. *Find the sum of the first $n+1$ coefficients in the expansion of* $(1-x)^{-3}$.

$$(1-x)^{-3} = 1 + 3x + \frac{3 \cdot 4}{1 \cdot 2}x^2 + \frac{4 \cdot 5}{1 \cdot 2}x^3 + \ldots + \frac{\overline{n+1} \cdot \overline{n+2}}{1 \cdot 2}x^n + \ldots.$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^n + \ldots.$$

Now, in multiplying together the two series on the right, we see that the coefficient of x^n is $1 + 3 + \dfrac{3 \cdot 4}{1 \cdot 2} + \ldots + \dfrac{\overline{n+1} \cdot \overline{n+2}}{1 \cdot 2}$.

$\therefore$ the required sum = coefficient of x^n in $(1-x)^{-4}$

$$= \frac{\overline{n+1} \cdot \overline{n+2} \cdot \overline{n+3}}{1 \cdot 2 \cdot 3}.$$

Examples. XLIX. f.

1. Prove that

$$\frac{\sqrt{3}}{2} = 1 - \frac{1}{2} \cdot \frac{1}{4} - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{1}{4^2} - \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{1}{4^3} - \frac{1 \cdot 1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6 \cdot 8} \cdot \frac{1}{4^4} - \ldots$$

2. Prove that $\dfrac{\sqrt{3}}{\sqrt{2}} = 1 + \dfrac{1}{6} + \dfrac{1 \cdot 3}{1 \cdot 2} \cdot \dfrac{1}{6^2} + \dfrac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} \cdot \dfrac{1}{6^3} + \ldots.$

3. Prove that the last series

$$=1+\frac{1}{2}\cdot\frac{1}{2}-\frac{1.1}{2.4}\cdot\frac{1}{2^2}+\frac{1.1.3}{2.4.6}\cdot\frac{1}{2^3}-\frac{1.1.3.5}{2.4.6.8}\cdot\frac{1}{2^4}+\dots.$$

[Since $(1-\frac{1}{3})^{-\frac{1}{2}}=\sqrt{1+\frac{1}{2}}$.]

Sum the series

4. $1+\frac{2}{3}+\frac{3}{3^2}+\frac{4}{3^3}+\frac{5}{3^4}+\dots$ to infinity.

5. $1+3\cdot\frac{2}{5}+\frac{3.4}{1.2}\cdot\left(\frac{2}{5}\right)^2+\frac{4.5}{1.2}\cdot\left(\frac{2}{5}\right)^3+\dots.$

6. Prove that $3\sqrt{2}=1+\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1.3}{2.4}\cdot\frac{1}{3^4}+\frac{1.3.5}{2.4.6}\cdot\frac{1}{3^6}+\dots.$

Find the sum of the first n coefficients in the expansion of

7. $(1-x)^{-4}$. 8. $(1-x)^{\frac{1}{2}}$. 9. $(1-x)^{-\frac{3}{2}}$. 10. $\frac{1+x}{(1-x)^2}$

11. Expand $(1-4x)^{\frac{1}{2}}$ to five terms. Show that the coefficient of x^n may be written $-\frac{2\,\underline{|2n-2}}{\underline{|n}\,\underline{|n-1}}$.

APPROXIMATIONS.

335. If a is a small fraction, a^2 is a small fraction of a.

Thus if a is *very* small, a^2 may be neglected in comparison with a in approximate calculations, or a^3 in comparison with a^2.

If we require a rough approximation we can reject all powers of a above the 1st; if a closer approximation is required we can retain the term containing a^2, and reject the higher powers; and so on according to the degree of accuracy required.

$$(1+a)^2=1+2a+a^2;$$

$\therefore$ if an approximate value of $(1+a)^2$ is required, we may call it $1+2a$.

$$(1+a)^3=1+3a+3a^2+a^3;$$

$\therefore$ $1+3a$ is an approximation for $(1+a)^3$.

A closer approximation is $1+3a+3a^2$.

In the same way,

$$(1+a)^n=1+na+\frac{n\,.\,\overline{n-1}}{1\,.\,2}a^2+\dots$$

$=1+na$ approximately, when a is a small fraction.

Also when α and β are both small fractions,

$$(1+\alpha)^m(1+\beta)^n=(1+m\alpha)(1+n\beta) \text{ approx.}$$
$$=1+m\alpha+n\beta \text{ approx.},$$

neglecting the product $mn\alpha\beta$, which is small compared with α and β.

Example 1. Calculate approximately $\sqrt[4]{623}$.

We know that $625=5^4$.

Now $\sqrt[4]{623}=(625-2)^{\frac{1}{4}}=625^{\frac{1}{4}}\left(1-\frac{2}{625}\right)^{\frac{1}{4}}$

$$=5\left\{1-\frac{1}{4}\cdot\frac{2}{625}+\frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{1\,.\,2}\left(\frac{2}{625}\right)^2-\dots\right\}$$
$$=5\{1-\tfrac{1}{4}(\cdot0032)-\tfrac{3}{32}(\cdot0032)^2-\dots\}$$
$$=5\{1-\cdot0008-\cdot00000096\} \text{ approximately}$$
$$=5\{1-\cdot00080096\}=4\cdot9959952.$$

Observing the 3rd term we notice that if we took only 2 terms the accuracy of the result to the first 4 places of decimals would not be affected. The result would then be 4·996.

Example 2. Find the value correct to 9 places of decimals of

$$(1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}} \text{ when } x=\cdot0003.$$

$$(1+2x)^{\frac{1}{2}}(1-3x)^{-\frac{1}{3}}=(1+x-\tfrac{1}{2}x^2+\tfrac{1}{2}x^3-\dots)(1+x+2x^2+\tfrac{14}{3}x^3+\dots)$$
$$=1+2x+\tfrac{5}{2}x^2+\tfrac{20}{3}x^3+\dots$$
$$=1+\cdot0006+\tfrac{5}{2}\times\cdot00000009+\tfrac{20}{3}\times\cdot000000000027+\cdots$$
$$=1\cdot000600225.$$

This is correct to 9 places, since the 4th term, if it were included, would not alter any of the first 9 places of decimals.

Example 3. If x be small compared with unity, find an approximate value of $\dfrac{(1+x)^{\frac{2}{3}}(1-2x)^{\frac{1}{2}}}{1-\frac{5}{6}x}$.

The expression $=\dfrac{(1+\frac{2}{3}x)(1-x)}{1-\frac{5}{6}x}$ if we expand the factors of the numerator by the Binomial Theorem and neglect x^2 and higher powers of x.

$\therefore$, by multiplication of the factors of the numerator,

$$\text{the expression}=(1-\tfrac{1}{3}x)(1-\tfrac{5}{6}x)^{-1}$$
$$=(1-\tfrac{1}{3}x)(1+\tfrac{5}{6}x)$$
$$=1+\frac{x}{2} \text{ approximately.}$$

Example 4. If x be small compared with unity,

$$\frac{(2+3x)^{\frac{1}{2}}(8+5x)^{\frac{2}{3}}}{(1-x)^{\frac{1}{4}}+(4-3x)^{\frac{1}{2}}}=\frac{2^{\frac{1}{2}}(1+\frac{3}{2}x)^{\frac{1}{2}}.8^{\frac{2}{3}}(1+\frac{5}{8}x)^{\frac{2}{3}}}{1-\frac{1}{4}x+4^{\frac{1}{2}}(1-\frac{3}{4}x)^{\frac{1}{2}}}$$
$$=\frac{4\sqrt{2}.(1+\frac{3}{4}x)(1+\frac{5}{12}x)}{1-\frac{1}{4}x+2(1-\frac{3}{8}x)}$$
$$=\frac{4\sqrt{2}.(1+\frac{7}{6}x)}{3-x}=\frac{4\sqrt{2}(1+\frac{7}{6}x)}{3\left(1-\frac{x}{3}\right)}$$
$$=\frac{4}{3}\sqrt{2}(1+\tfrac{7}{6}x)\left(1-\frac{x}{3}\right)^{-1}$$
$$=\frac{4}{3}\sqrt{2}(1+\tfrac{7}{6}x)(1+\tfrac{1}{3}x)$$
$$=\frac{4}{3}\sqrt{2}(1+\tfrac{3}{2}x)\text{ approximately.}$$

A closer approximation would be obtained by neglecting only powers of x above the 2nd.

Example 5. If l feet be the length of a pendulum which beats seconds, how many beats will it lose in 24 hours if it be lengthened one per cent., the time of a beat in seconds being given by the expression $\pi\sqrt{\frac{l}{32}}$?

Let t be the time (in seconds) of the beat of the lengthened pendulum.

Then $t=\pi\sqrt{\frac{l}{32}\cdot\frac{101}{100}}$.

But $1=\pi\sqrt{\frac{l}{32}}$ (since the original pendulum beats seconds).

$\therefore$ by division $t=\sqrt{1+\frac{1}{100}}$.

The number of beats in 24 hours = the number of seconds in 24 hrs. ÷ the time of 1 beat $=\dfrac{24\times60\times60}{(1+\frac{1}{100})^{\frac{1}{2}}}=86400(1+\frac{1}{100})^{-\frac{1}{2}}$

$=86400(1-\frac{1}{2}\cdot\frac{1}{100})$ approximately

$=86400(1-\frac{1}{200})=86400-432.$

It originally did make 86400 beats;

$\therefore$ the number of beats lost = 432.

336. *Approximate division. Divide* 5758 *by* 997.

$5758\div997=5758\div(1000-3)$

$=5758\div1000(1-\frac{3}{1000})=5{\cdot}758(1-\frac{3}{1000})^{-1}$

$=5{\cdot}758\times(1+\frac{3}{1000})$ approximately

$=5{\cdot}758+\frac{3}{1000}$ of $5{\cdot}758$

$\doteq5{\cdot}758+{\cdot}017\ldots$

$=5{\cdot}775,$

a result which is correct to 3 decimal places.

337. Approximation for the r^{th} root of a number.

If $\sqrt[r]{\mathrm{N}}=a+x$, where a is an approximation and consequently x is small, then a closer approximation is given by the formula

$$\sqrt[r]{\mathrm{N}}=\frac{(r+1)\mathrm{N}+(r-1)a^r}{(r-1)\mathrm{N}+(r+1)a^r}\cdot a. \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

For instance, in finding a cube root,

$$\sqrt[3]{\mathrm{N}}=\frac{4\mathrm{N}+2a^3}{2\mathrm{N}+4a^3}\cdot a=\frac{2\mathrm{N}+a^3}{\mathrm{N}+2a^3}\cdot a.$$

To find $\sqrt[3]{2}$.

Here an approximation is $\frac{5}{4}$.

$$\therefore\ \sqrt[3]{2}=\frac{2\times 2+\frac{125}{64}}{2+\frac{250}{64}}\cdot\frac{5}{4}=\frac{1905}{1512}=1{\cdot}25992\ldots .$$

This approximation gives the first 5 decimal places of $\sqrt[3]{2}$ correctly.

If $p-q$ be small compared with p or q,

$$\sqrt[r]{\frac{p}{q}}=\frac{(r+1)p+(r-1)q}{(r-1)p+(r+1)q} \text{ approximately.}$$

For $$\sqrt[r]{\frac{p}{q}}=\frac{\{(p+q)+(p-q)\}^{\frac{1}{r}}}{\{(p+q)-(p-q)\}^{\frac{1}{r}}}$$

$$=\frac{\{1+(p-q)/(p+q)\}^{\frac{1}{r}}}{\{1-(p-q)/(p+q)\}^{\frac{1}{r}}}=\frac{1+\frac{1}{r}(p-q)/(p+q)}{1-\frac{1}{r}(p-q)/(p+q)} \text{ nearly}$$

$$=\frac{(r+1)p+(r-1)q}{(r-1)p+(r+1)q}.$$

This becomes the formula given above, if $p=\mathrm{N}$ and $q=a^r$.

Examples. XLIX. g.

1. Write down the value of $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$ when $x=10$; and find its value, to the nearest hundredth, when $x=10{\cdot}1$.

2. Find the value, to the nearest hundredth, of $\left(x+\frac{1}{x}\right)^4$ when $x=10$. Also find the value of the same expression, to the nearest hundredth, when $x=10{\cdot}1$.

3. Find the value of $\left(x-\frac{1}{x}\right)^3$ when $x=10$. Also find its value, to the nearest hundredth, when $x=9{\cdot}9$.

4. Test the following rules for approximation:

(i) To divide by π, multiply by $\frac{1}{3}-\frac{1}{100}-\frac{1}{200}$.

$\left(\text{Compare this expression with the value of } \frac{1}{\pi}.\right)$

(ii) To divide by π^2, multiply by $\frac{1}{10}+{\cdot}0013$.

5. Find the value of $1{\cdot}0001^5$ correct to 4 places of decimals.

6. .... $1{\cdot}0006^{\frac{1}{2}}$ .... 5 places.

7. .... $\dfrac{1}{(1-\frac{1}{8}x)^2}$ .... 8 places when $x={\cdot}001$.

8. $(1+x)^3(1-x)^{-3}$ .... 3 places when $x={\cdot}0002$.

9. . $\sqrt[3]{1005}$ 3 places.

10. . $\sqrt{99}$ 5 places.

11. . $\sqrt[5]{31}$

12. . $\sqrt[3]{510}$

13. . $\sqrt[3]{28}$

14. . $\sqrt[5]{3123}$

15. The side of a square is increased by ·1 per cent. What percentage of increase is there in the area?

16. When x is numerically <1, prove that $1-x$ is an approximate value of $\dfrac{1}{1+x}$, and that $1+x$ is an approximate value of $\dfrac{1}{1-x}$. What is the amount of the error when $x=\frac{1}{100}$?

17. Find the sq. rt. of $1+x$ as far as the term in x^2. [x is numerically <1.]

18. Approximate to $\sqrt{1{\cdot}03}$.

19. If we use the formula $(1-x)^{\frac{1}{2}}=1-\frac{1}{2}x$ to find $\sqrt{99{\cdot}7}$, to how many decimal places is it correct?

Use the Binomial Theorem to find the value of

20. $3457\div998$ correct to 3 decimal places.

21. $7{\cdot}821\div1{\cdot}002$ ....

22. $4831\div996$ ....

23. $341\div99^2$ ....

24. $9381675\div999^2$ ....

25. $5{\cdot}231\div(0{\cdot}996)^2$ ....

26. The edge of a cube is diminished by ·01 per cent. What percentage of the original volume is the decrease of volume?

27. A pendulum which beats seconds is lengthened by ·04 per cent. of its length. How many beats does it lose in 24 hours?

[Time of 1 beat $=\pi\sqrt{\dfrac{l}{32}}$.]

28. If x is so small that powers above x^2 may be neglected, find the value of $(1+7x)^8-(1+8x)^7$.

When x is very small, prove that

29. $\dfrac{1}{(1-3x)^2(1-2x)^3}=1+12x$ approximately.

30. $\dfrac{1+x}{(1-4x)^5}=1+21x$ approximately.

31. $\dfrac{(1+x)^{\frac{1}{2}}}{(1+x)^{\frac{1}{3}}}=1+\dfrac{x}{6}$ approximately.

32. $\dfrac{(4+x)^{\frac{1}{2}}}{(3-x)^2}=\dfrac{2}{9}\left(1+\dfrac{19x}{24}\right)$ approximately.

33. $\dfrac{(1-x)^{\frac{1}{2}}+(1+x)^{\frac{2}{3}}}{(1-x)+(1-x)^{\frac{1}{2}}}=1+\dfrac{5x}{6}$ approximately.

34. $\dfrac{(1+x)^{\frac{1}{2}}+(1-x)^{\frac{1}{4}}}{(1-x)^{\frac{1}{2}}+(1+x)^{\frac{1}{3}}}=1+\dfrac{5x}{24}$ approximately.

35. $\dfrac{\left(\dfrac{9x}{2}+1\right)^{\frac{1}{2}}(1-3x^2)^{\frac{1}{3}}}{\left(1+\dfrac{9x}{8}\right)^2}=1-\dfrac{307}{64}x^2$ approximately.

36. Expand $\left(1-\dfrac{a}{D}\right)^{-1}$ in powers of a to the term in a^3. When a balance is used in air of density a, and a body of density D is balanced in it by brass weights of density d weighing m grams, the real weight of the body in grams is

$$m\left(1-\frac{a}{d}\right)\Big/\left(1-\frac{a}{D}\right).$$

Expand this expression to the first power of a, and find how much the weight of the body differs from that of the brass weights when $m=1000$, $a=\cdot 0012$, $d=8$, $D=\cdot 5$.

37. Find the coefficient of x^8 in the expansion of $\dfrac{1}{1+x+x^2+x^3}$.

38. If $N=(a+x)^3$, where x is small in comparison with a, prove that

$$\sqrt[3]{N}=\frac{2N+a^3}{N+2a^3}\cdot a \text{ approximately.}$$

CHAPTER L.

REVISION PAPERS.

L. a.

1. Find all the factors of $42x^3+17x^2-8x-3$, given that $2x+1$ is one of them.

2. Simplify $\dfrac{2+\sqrt{3}}{4+\sqrt{3}}+\dfrac{4-2\sqrt{3}}{5-2\sqrt{3}}$.

3. If α, β be the roots of $x^2+px+q=0$, find the equation whose roots are $m\alpha^2+n\beta^2$ and $m\beta^2+n\alpha^2$.

4. Find the value of $1{\cdot}04^{-20}$ correct to 2 decimal places.

5. Prove, without assuming the value of ${}^nC_r$, that ${}^nC_r=\dfrac{n}{r}\,{}^{n-1}C_{r-1}$.

6. Find the greatest coefficient in the expansion of $(1+x)^{12}$.

7. Draw the graphs of $y=2x+\frac{1}{2}$ and $y^2=4x$, and so solve the simultaneous equations.

L. b.

1. If $a:b=b:c$, each ratio$=b^2/(a+b):c^2/(b+c)$.

2. In how many ways can a crew of 8 be arranged in a boat, if only 3 are fit to row the stroke oar?

3. A man walking from A to B at the rate of $3\frac{1}{2}$ miles an hour starts 40 minutes before the departure of a coach from A that goes 10 miles an hour, and is picked up by it on the way. When he arrives at B, he finds that his journey by coach has lasted $2\frac{2}{3}$ hours. Find the distance from A to B.

4. If $\log_4 N={\cdot}35184$, find $\log_8 N$.

5. The first and last of an odd number of quantities in H.P. are $\frac{3}{2}-\sqrt{2}$ and $\frac{3}{2}+\sqrt{2}$: find the middle term.

6. Find the conditions which must be satisfied in order that the equations $x^2+ax+b^2=0$, $x^2-bx+a^2=0$, may have (1) the same roots, (2) one root common.

7. Find the middle term in the expansion of $\left(x+\dfrac{1}{2x}\right)^{10}$.

L. c.

1. The roots of the equation $ax^2+2bx+c=0$ will be imaginary if a, b, c are in H.P. and have the same sign.

2. Between two numbers whose sum is $2\frac{1}{6}$ an even number of arithmetic means is inserted. The sum of these means exceeds their number by 1. How many means are there?

3. Ten men are to be selected from 4 companies, a different number being taken from each company, and no company being without a representative. How many different arrangements can be made as to the different numbers to be taken from the several companies?

4. If one-tenth of the trees in a plantation are cut down every year, after how many years will less than one-third of the original number be left? [$\log 3 = \cdot 4771$.]

5. Find the 10^{th} term in the expansion of $\left(x - \frac{1}{x}\right)^{15}$.

6. Find $\sqrt[5]{\cdot 002}$, using logarithms.

7. The natural numbers are divided into groups thus:

$$1;\ 2, 3, 4;\ 5, 6, 7, 8, 9;\ \text{etc.}$$

Prove that the sum of the numbers in the n^{th} group is $n^3 + (n-1)^3$.

L. d.

1. Express $\cdot\dot{7}$ in the scale of 9 and test by summing a G.P.

2. Given $\log_{10} 5 = \cdot 69897$, find to 3 places $\log \sqrt{\cdot 08}$ to the base 125.

3. 864 bricks each containing $110 \cdot 592$ cub. inches, are piled in a stack, whose length and breadth are each double of its height. What is the height?

4. Find the value of $\sqrt{(3 + 8\sqrt{7 + 4\sqrt{3}})} - \sqrt{3}$.

5. The volume of a gas varies as the absolute temperature and inversely as the pressure; and with temperature 300 and pressure $13 \cdot 5$ the volume is 120 c. inches. What is the volume when the temperature is 350 and pressure 15?

6. Find the $\overline{r+1}^{\text{th}}$ term in the expansion of $\left(3 - \frac{x}{3}\right)^{-3}$.

7. Find $\sqrt[5]{99999 \cdot 5}$ correct to 11 decimal places.

L. e.

1. Sum to n terms the series whose n^{th} term is $(n+1)(n+3)$.

2. What number of 2 digits is that which, if its digits be reversed, becomes 1 less than its half, the sum of the digits being 7?

3. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, each of these fractions $= \left(\frac{a^4 + c^4 + e^4}{b^4 + d^4 + f^4}\right)^{\frac{1}{4}} = \left(\frac{ace}{bdf}\right)^{\frac{1}{3}}$.

4. Transform $4569 \cdot 246$ from the ordinary scale to the scale of 5, obtaining a correct result to 4 places of fifths.

5. Prove that the greatest number of combinations that can be formed with $2n$ things, each combination containing the same number of things, is double the greatest number that can be formed with $2n - 1$ things.

6. Find the 5th term in the expansion of $\left(2x - \frac{x^2}{2}\right)^{\frac{9}{2}}$ and the coefft. of x^r in the expansion of $(1-x)^{-\frac{1}{2}}$.

7. If a pendulum of length l feet, which beats seconds, be lengthened one half per cent., find the number of beats in 24 hours, the time of a beat being $\pi\sqrt{l \div 32}$ seconds.

L. f.

1. Find an expression, of the 2nd degree in x, which is equal to $2, -1, 34$, when $x=1, -2, 3$.

2. Express $2(a^2+b^2+c^2-bc-ca-ab)$ as the sum of three squares.

3. Find by logarithms (i) $\sqrt[7]{2\cdot718}$, (ii) $\cdot296\times3\cdot378$, (iii) e^π, where $e=2\cdot718$ and $\pi=3\cdot14$.

4. If a, b, c are in G.P., and $a-b$, $c-a$, $b-c$ in H.P., prove that $a+4b+c=0$.

5. Show that $-1^2+2^2-3^2+4^2-\ldots$ to $2n$ terms $=1+2+3+\ldots+2n$.

6. The square of the sum of two numbers is divided by the product of the sum of their cubes and the difference of their squares. Express the result algebraically in lowest terms.

7. A parent puts in a box for a child on every birthday a half-crown for every year of its age. How old will the child be when the whole of the money in the box is £17?

L. g.

1. Solve
$$\left.\begin{aligned}3x^2+5xy&=22\\11xy-3y^2&=19\end{aligned}\right\}.$$

2. Trace the values of $x+\frac{1}{x}$ as x changes from -3 to $+3$, and represent it graphically. Find its least numerical value.

3. Sum the infinite series $\frac{1}{1\cdot05}+\frac{1}{1\cdot05^2}+\frac{1}{1\cdot05^3}+\ldots.$

4. The sum of the squares of the first n natural numbers is $20n$. Find n.

5. How many sums can be made with the following coins: a penny, a sixpence, a shilling, a half-crown, a crown, a sovereign?

6. Represent graphically $11+6\sqrt{2}$. Find its square root graphically and test your accuracy by finding the result algebraically.

7. If a square number ends in 9, the preceding digit is even.

L. h.

1. A series whose 1st, 2nd, and 3rd terms are
$$\frac{1}{\sqrt{2}},\ \frac{1}{1+\sqrt{2}},\ \frac{1}{4+3\sqrt{2}}$$
is either Arithmetic or Geometric. Determine which it is, and write down the 4th term.

2. Find the sq. rt. of $3(x-1)+2\sqrt{2x^2-7x-4}$.

3. Find the value of x from the equation $18^{8-4x}=(54\sqrt{2})^{3x-2}$, using as base $3\sqrt{2}$.

4. Prove that, if two numbers have the same digits, their difference is a multiple of 9.

5. In how many ways can 10 persons be divided into parties of 3 and 7? In how many ways can 10 things be divided between 2 persons so that one may have 3, and the other 7?

6. Find the greatest coefficient in the expansion of

$$\text{(i) } (1+x)^{21}, \quad \text{(ii) } (1+x)^{24}.$$

7. Find the coefficient of x^r in the expansion of $(1+2x)/(1-x)^3$.

L. k.

1. A diagonal of a cyclic quadrilateral is given by the expression

$$\left\{\frac{(bc+ad)(ac+bd)}{ab+cd}\right\}^{\frac{1}{2}},$$

where a, b, c, d are the sides taken in order. If the sides, in order, are 60, 25, 52, 39, prove that the diagonals are 65 and 63, and the area is 1764. [Divide into 2 triangles.]

2. The value of a lease is found to be £6000$(1-1{\cdot}04^{-60})$. Calculate it by logarithms.

3. Sum to n terms the series $1^3+3^3+5^3+\dots$.

4. What fraction in the common scale is equal to $\cdot 5\dot{3}$ in the scale of 7?

5. If a, b, c are in H.P., then

$$\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)+\frac{\;}{b^2}=\frac{\;}{ac}.$$

6. Determine which is the greatest term in the expansion of $(3+5x)$ when $x=\frac{1}{2}$.

7. Show how to sum the series $1+2x+3x^2+\dots+nx^{n-1}$; prove that

$$1+2\left(1+\frac{1}{n}\right)+3\left(1+\frac{1}{n}\right)^2+\dots+n\left(1+\frac{1}{n}\right)^{n-1}=n^2.$$

L. l.

1. If one of the equations $x^2-x(3c-b)+bc=0$ and $x^2-x(5c-b)+4c^2=0$ has equal roots, so has the other.

2. If 1, x, y are in A.P., and 1, y, x in G.P., find x and y.

3. How many words, each of 7 letters, can be formed from 3 vowels and 4 consonants, in which no two consonants are together?

4. Find log $\cdot 00132874$ to 6 decimal places, given that

$$\log 1{\cdot}3287=\cdot 123427, \quad \log 1{\cdot}3288=\cdot 123460.$$

5. The diameter of the earth being 7900 miles and that of the moon 2160, find the diameter of a sphere whose surface is the sum of their surfaces. (Surface $\propto$ square of diameter.)

6. If $(1+x)^n=c_0+c_1x+c_2x^2+\dots+c_nx^n$, show that

$$2c_0+2^2\frac{c_1}{2}+2^3\frac{c_2}{3}+\dots=\frac{3^{n+1}-1}{n+1}.$$

7. Each of two bags contains 12 different coins; find how many different combinations of 10 coins can be made by taking 5 out of each bag.

L. m.

1. If a, b, c, d are in G.P., prove that

$$(a^2+ac+c^2)(b^2+bd+d^2)=(ab+bc+cd)^2.$$

2. What number added to the expression

$$\frac{x^2}{9}+\frac{1}{4x^2}+\frac{2}{3}x+\frac{1}{x}$$

will make it a square?

3. Sum to infinity $\frac{1}{\sqrt{2}}+\frac{1}{1+\sqrt{2}}+\frac{\sqrt{2}}{3+2\sqrt{2}}+\ldots$.

4. In the equation $x^2-ax+\frac{a^2-b^2}{4}=0$ find the value of b when the roots are equal.

5. A walks m miles in n hours; B walks $6n$ miles in $\frac{1}{3}m$ hours; and the difference of their rates of walking is $\frac{1}{2}$ mile per hour. Find the rate at which each walks.

6. Show that $(x^2-bc)(2x-b-c)^{-1}$ has no real value between b and c.

7. Solve the equation $a^x(a^x-1)=2$; and find the value of

$$10\log\tfrac{3}{2}+7\log\tfrac{5}{18}+4\log\tfrac{48}{5}.$$

L. n.

1. The product of any 4 consecutive even integers increased by 16 is a perfect square.

2. If $x=2+\sqrt{2}$, then $x^2+\frac{4}{x^2}=12$.

3. Find the relation between a, b, c when $x+y=a$, $x^2+y^2=b$, $x^3+y^3=c$.

4. Solve $\left.\begin{array}{ll} 2(x^4+y^4)+7xy(x^2+y^2)=740 \\ 2(x^2+y^2)-xy \qquad\quad =20 \end{array}\right\}.$

5. Ten men are chosen in every possible way out of 16. In how many of the groups do 2 particular men occur?

6. Find the coefficient of x^r in the expansion of $(1+x)(1-x)^{-4}$.

7. Find which is the greatest term in the expansion of $(1-x)^{-\frac{4}{3}}$ when $x=\frac{12}{13}$.

L. p.

1. If $x=3-\sqrt{3}$, then $x^2+\frac{36}{x^2}=24$.

2. If a, b, c are in A.P., and $a, b, c+1$ are in G.P., prove that

$$a=(a-b)^2=(b-c)^2.$$

3. A number consists of 3 digits in G.P. The sum of the right-hand and left-hand digits exceeds twice the middle digit by 1; and the sum of the left-hand and middle digits is two-thirds of the sum of the middle and right-hand digits. Find the number.

4. Prove ${}^{2n+4}C_{n+2} + {}^{2n+4}C_{n+3} = {}^{2n+5}C_{n+3}$.

5. Sum the series $1+2\times 6+3\times 6^2+4\times 6^3+5\times 6^4$, by expressing it first in the scale of 6 and then converting it.

6. $105^x = 100$. Find x to 4 decimal places.

7. Expand $(1-x)^{-4}$ to 5 terms, and write down the $(r+4)^{\text{th}}$ term in its simplest form.

Apply the Binomial Theorem to prove that

$$1+\frac{3}{4}+\frac{3.5}{4.8}+\frac{3.5.7}{4.8.12}+\dots \textit{ ad inf.} = \sqrt{8}.$$

L. q.

1. A merchant at the end of the 1st year had doubled his original capital, the 2nd year he gained £80 more than the square root of his increased capital, the 3rd year he cleared half the square of all that he had at the end of the 2nd, and found himself with £18240. How much had he at first?

2. The common difference of an A.P. is $-d$, and the sum of n terms is $\frac{(2a+d)^2}{9d}$; find n.

3. Prove that $2^{\frac{1}{3}} = 2^{\frac{1}{4}}.2^{\frac{1}{16}}.2^{\frac{1}{64}} \dots$ *ad inf.*, and hence show that it is possible to find cube root by continued extractions of square root and multiplication. Find an approximate value of $2^{\frac{1}{3}}$ by two multiplications, and show to how many figures it is correct.

4. Prove that $2^{4n}-1$ is divisible by 15.

5. The equation $y=mx+c$ when combined with $y^2=4ax$ gives equal roots for x and equal roots for y. Find c.

6. By the Binomial Theorem, sum the series

$$1+\frac{1}{2^2}+\frac{1.3}{\underline{|2}}\cdot\frac{1}{2^4}+\frac{1.3.5}{\underline{|3}}\cdot\frac{1}{2^6}+\dots \textit{ ad inf.}$$

7. Find the coefft. of x^r in $(1+2x+3x^2+4x^3+\dots \textit{ ad inf.})^n$, where x is numerically < 1.

L. r.

1. Simplify $7\log\frac{16}{15}+5\log\frac{25}{24}+3\log\frac{81}{80}$.

2. If a, b, c are in H.P., $a^2+c^2 > 2b^2$.

3. If $x+y+z=a$, and $x^2+y^2+z^2=b^2$, express $x^3+y^3+z^3-3xyz$ in terms of a and b.

4. Solve the equation $1+\frac{1}{2}x+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\dots \textit{ ad inf.} = \sqrt{1{\cdot}25}$.

5. Find the first negative term in the expansion of $(1+x)^{\frac{7}{2}}$, x being positive.

6. Find 5 numbers in A.P. whose sum shall be 25 and their product 2520.

7. In how many ways can a party of 6 be selected from 20 people, so that the party may never contain more than one of two specified?

L. s.

1. If $f(n)=6n^2+5n+1$, find $\sum\limits_1^n\{f(n)\}$.

2. Find the number of diagonals of a polygon of 16 sides.

3. The expenses of a house are partly fixed and partly $\propto$ the number of inmates. When the numbers are 8 and 9, the expenses are £1020 and £1100. Find the fixed part.

4. Obtain $\sqrt[3]{510}$ approximately by the Binomial Theorem, and check your result by logarithms.

5. On a division in the House of Commons, if the number of members for the motion had been increased by 50 from the other side, the motion would have been carried by 5 to 3; but if those against the motion had received 60 from the other party, the motion would have been lost by 4 to 3. Did the motion succeed, and how many members voted?

6. Three consecutive coefficients in the expansion of $(1+x)^n$ are 91, 364 and 1001. Find n.

7. Find the sum of the first r coefficients in the expansion of $(1-x)^{-5}$.

L. t.

1. How many numbers less than 1000 are there which contain the digit 3 at least once?

2. If x is very small, $\dfrac{\sqrt{1+x}+(1-3x)^{\frac{1}{3}}}{\sqrt{1-x}+(1+2x)^{\frac{1}{3}}}=1-\dfrac{5}{24}x$ nearly.

3. How may a body of 754 lbs. be weighed with weights of 1, 3, 3^2, 3^3 ... lbs., only one of each being used?

4. Draw the graph of $y^2=12x$, and find where it is met by the line

$$y=2x+\tfrac{3}{2}.$$

5. Prove that $3^{4n}-1$ is divisible by 80.

6. By logarithms, calculate the square root of the reciprocal of 6241.

7. If $f(x)=\dfrac{a^x-a^{-x}}{a^x+a^{-x}}$, prove that $f(x+y)=\dfrac{f(x)+f(y)}{1+f(x)f(y)}$.

CHAPTER LI.

INTEREST AND ANNUITIES.

338. Simple Interest. If r be the interest on £1 for 1 year, Pr is the interest on P pounds for 1 year.

$\therefore$ the interest on P pounds for n years is Pnr.

If V be the present value of P due n years hence, V with its interest must amount to P in n years.

$$\therefore\ V + Vnr = P; \quad \therefore\ V = \frac{P}{1+nr}.$$

$$\text{Discount} = P - V = P - \frac{P}{1+nr} = \frac{Pnr}{1+nr}.$$

This is called *true discount*. In actual practice the discount allowed is *simple interest*.

339. In **Compound Interest** the Principal does not remain constant, but is increased by the addition of the Interest whenever the latter is paid.

If I be the interest on P, on which the interest is paid annually, the principal for the 2nd year is $P + I$.

340. *If £P amount to £M in n years, to prove that* $M = PR^n$, *where* R *is the amount of £1 in one year.*

In 1 year £1 amounts to £R.

∴ .... £P .... £PR.

In the 2nd year this PR amounts to PR . R.

∴ in 2 years P amounts to PR^2.

In 3 years P amounts to PR^3; and so on.

∴ the amount of P in n years at compound interest is PR^n.

341. If V is the present value of P due n years hence, reckoning compound interest,

V with its interest must amount to P in n years.

$$\therefore\ VR^n = P; \quad \therefore\ V = P \cdot R^{-n}$$

342. If interest is at the rate of r per cent.,

$$\text{the interest on £1 for 1 year} = \frac{r}{100},$$

$$\text{and the amount of £1 in 1 year} = 1 + \frac{r}{100}.$$

Thus, at 5 %, the interest on £1 for 1 year = £·05,

and the amount of £1 in 1 year = £1·05.

Example. Find, as accurately as you can with 4-figure logarithm tables, the amount of £500 at 4 % in 20 years, at compound interest.

Using the formula $M = PR^n$, we have

$$P = 500, \quad R = 1{\cdot}04, \quad n = 20.$$

$$\therefore \; M = 500(1{\cdot}04)^{20}.$$

$$(1{\cdot}04)^{20} = \text{antilog}\,(20 \times {\cdot}0170) = \text{antilog}\,({\cdot}340)$$
$$= 2{\cdot}188.$$

$$\therefore \; M = 500 \times 2{\cdot}188 = \frac{2188}{2} = £1094.$$

343. In some cases compound interest is payable more frequently than once a year; *e.g.* it may be required *to find the amount of £1000 in 2 years at 6 per cent. per annum, the interest being paid each half-year.*

Here the interest for the 1st half-year $= \frac{1}{2}$ of $\frac{6}{100}$ of £1000 $=$ £30. This is added to the principal, making it £1030; and the interest for the 2nd half-year $= \frac{1}{2}$ of $\frac{6}{100}$ of 1030; and so on.

Thus the process is equivalent to calculating the amount for twice as many years at half the given rate.

If r be the simple interest on £1 for 1 year, the amount of £P at the end of the 1st half-year $= P + \dfrac{Pr}{2} = P\left(1 + \dfrac{r}{2}\right)$.

Thus each half-year the sum of money is multiplied by $\left(1 + \dfrac{r}{2}\right)$.

$\therefore$ the amount at the end of n years, *i.e.* at the end of $2n$ half-years $= P\left(1 + \dfrac{r}{2}\right)^{2n}$.

If the interest were paid *quarterly* at the same rate, the result would be the same as if the rate were one quarter of what it is, and the payments were made annually for four times as many years.

The amount at the end of n years $= P\left(1 + \dfrac{r}{4}\right)^{4n}$.

344. The **Compound Interest Law** is applicable to questions of population.

Example 1. In a town whose population is p, the increase is 2 per cent. annually. What will the population be in n years?

At the end of the 1st year,

$$\text{the population} = p + p \times \tfrac{2}{100} = p\left(1 + \tfrac{2}{100}\right) = p \times 1{\cdot}02.$$

The next year *this* population is multiplied by the same quantity, $1\cdot02$.

$\therefore$ the population at the end of the 2nd year $= p \times 1\cdot02^2$.

.... 3rd ...... $= p \times 1\cdot02^3$.

.... nth ...... $= p \times 1\cdot02^n$.

Example 2. If the increase of population in a town is 7 per thousand annually, in how many years will the population be doubled?

In n years the population, originally P, becomes

$$P \times (1 + \tfrac{7}{1000})^n, \quad i.e. \; P \times 1\cdot007^n.$$

$$\therefore \; P \times 1\cdot007^n = 2P$$

$$\therefore \; 1\cdot007^n = 2.$$

$$\therefore \; n \log 1\cdot007 = \log 2.$$

$$\therefore \; n = \frac{\log 2}{\log 1\cdot007} = \frac{\cdot3010}{\cdot0029} = 103\cdot8 \text{ approximately.}$$

$\therefore$ we may say that the town doubles itself in population in rather less than 104 years.

Examples. LI. a.

1. At what rate per cent. simple interest will £91. 13*s.* 4*d.* amount to £100 in 4 years?

2. In what time will £315 become £378 at 4 per cent. simple interest?

3. Find the true discount on £720 due 3 months hence at $2\frac{1}{2}$ per cent. simple interest.

4. Find the compound interest on £620 for 5 years at 3 p.c.

5. .... £50 ...... 10 .... 4 .... .

6. Find at compound interest the amount of £825 in 6 yrs. at $3\frac{1}{2}$ p.c.

7. .... £120. 17*s.* 6*d.* ... 7 .. .. 4 .....

8. .... £30. 5*s.* 10*d.* ... 14 ... . 5 .....

9. In what time does a sum of money double itself at 5 p.c. compound interest?

10. In what time at 4 p.c.?

11. Find the amount of £10 in 40 years at 5 p.c. compound interest.

12. If the birth rate in a place be 76 in 1000, the death rate 48 in a thousand, in how many years will the population be doubled?

13. **At simple interest, the interest minus the discount = the interest on the discount.**

14. At what rate, simple interest, will the interest on £326 for $7\frac{1}{2}$ years be £110. 0*s.* 6*d.*?

15. In what time will £502. 13*s.* 4*d.* amount to £578. 1*s.* 4*d.* at $4\frac{1}{2}$ per cent. simple interest?

16. The time is 3 months, rate 4 p.c., simple interest £5. 10*s.* $10\frac{4}{5}$*d.*; what is the principal?

17. The rate is $4\frac{1}{2}$ p.c., simple interest £$5\frac{5}{8}$, principal £380. 4*s.* 2*d.*; find the time.

18. Find the amount of £6000 in 6 yrs. at 5 p.c. compound interest.

19. The present value of £826. 10s. due $3\frac{1}{2}$ years hence at simple interest is £725. Find the rate.

20. The discount on £$618\frac{1}{8}$ at 4 p.c. is £$80\frac{5}{8}$. Find the time at which it is due.

21. The difference between interest and discount at $2\frac{2}{3}$ p.c. for $2\frac{1}{2}$ years is £2. 11s. What is the sum due?

22. In what time will a given sum treble itself at 3 p.c. per annum, compound interest being payable half-yearly?

23. Three sums in A.P. are put out to compound interest for 2, 3, 4 years at 5%. The amounts are then in A.P., but the sum which was greatest is now least. Find the ratio of the greatest to the least.

24. The interest on a certain sum of money for 9 months is £30. 18s., and the discount at the same rate is £30. Find the sum of money and the rate.

25. Calculate in decimal form the terms of $\left(1+\frac{3}{100}\right)^5$. Find the compound interest on £100 for 5 yrs. at 3 per cent.

26. Money is put out to compound interest at $2\frac{1}{2}$ per cent.; show that it will have more than doubled itself in 29 years.

27. A sum of £1000 is lent on condition that it shall bear compound interest at 5 p.c. per ann. for the first 5 years, 10 p.c. per ann. for the next 5 years, and afterwards 15 p.c. per ann. What will the debt amount to in 20 years?

28. The number of births in a country being annually 58 per 1000 inhabitants and the loss from deaths, etc., 25 per 1000, find in how many years the population will be doubled.

29. The annual excess of births over deaths in a country is 15 per 1000. Find the number of years required for the population to be doubled.

30. The births in a town are 42 per 1000 annually, and the loss through death and other causes 17 per 1000. In how many years will the population be half as much again?

ANNUITIES.

345. An *Annuity* is a series of equal annual payments.

An Annuity may be *terminable*, *i.e.* it may cease at the end of a certain number of years; or it may be *perpetual*.

In calculating the accumulation of an annuity which has remained unpaid for a number of years, or the present value of an annuity, the question of interest comes in. In such questions simple interest is inapplicable, as it gives contradictory results. In the following investigations, therefore, it is assumed that *compound interest* is implied.

One form of annuity of £A is a property bringing in £A annually, such as a **freehold** estate bringing in rents to that annual amount. If the present value of such a property, *i.e.* the sum of money required to buy it, is £nA, the property is said to be worth *n years' purchase.*

A **leasehold** estate is one which is held for a limited number of years; and the income from it may be regarded as a *terminable* annuity.

When an estate is bound to come into a man's possession after a certain number of years, that man is said to have a **reversionary interest** in it; and the present value of the **reversion** is the present value of the income which will begin after so many years.

346. *To find the present value of an annuity to continue for n years.*

Let the annuity be £A, r the interest on £1 for a year.

$$\therefore \text{ amount of £1 in 1 year} = 1 + r = \mathsf{R}.$$

The present value of the 1st payment $= \dfrac{\mathsf{A}}{\mathsf{R}}$, (Art. 341)

$$\ldots\ldots\ldots\ldots\ldots\ldots \text{ 2nd } \ldots\ldots\ldots = \frac{\mathsf{A}}{\mathsf{R}^2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots \text{ 3rd } \ldots\ldots\ldots = \frac{\mathsf{A}}{\mathsf{R}^3}, \text{ and so on.}$$

$\therefore$ the present value of the annuity $= \dfrac{\mathsf{A}}{\mathsf{R}} + \dfrac{\mathsf{A}}{\mathsf{R}^2} + \dfrac{\mathsf{A}}{\mathsf{R}^3} + \ldots$ to n terms

$$= \frac{\mathsf{A}}{\mathsf{R}} \cdot \frac{1 - \mathsf{R}^{-n}}{1 - \mathsf{R}^{-1}} = \frac{\mathsf{A}(1 - \mathsf{R}^{-n})}{\mathsf{R} - 1}.$$

When $n = \infty$, R^n is infinite, and R^{-n} is indefinitely small; $\therefore$ the present value of a perpetual annuity

$$= \frac{\mathsf{A}(1 - \mathsf{R}^{-n})}{\mathsf{R} - 1} \text{ where } n = \infty$$

$$= \frac{\mathsf{A}}{\mathsf{R} - 1} = \frac{\mathsf{A}}{r}.$$

NOTE.—The present value (P) of a perpetual annuity might also be found as follows.

The annual interest on P must be just what is required to pay the annuity A. But the interest on P $= \mathsf{P}r$.

$$\therefore \mathsf{P}r = \mathsf{A}. \quad \therefore \mathsf{P} = \frac{\mathsf{A}}{r}.$$

347. *To find the present value of an annuity of £A to begin m years hence and to continue for p years.*

The first payment is to be made at the end of $m+1$ years.

$$\therefore \text{ the present value of the 1st payment} = \frac{A}{R^{m+1}};$$

$$\text{2nd } \ldots\ldots \quad = \frac{A}{R^{m+2}}.$$

the present value of the annuity

$$= \frac{A}{R^{m+1}} + \frac{A}{R^{m+2}} + \ldots \text{ to } p \text{ terms}$$

$$= \frac{A}{R^{m+1}} \cdot \frac{1 - \frac{1}{R^p}}{1 - \frac{1}{R}} = \frac{A}{R^m} \cdot \frac{1 - R^{-p}}{R - 1}$$

$$= \frac{A}{r} \cdot (R^{-m} - R^{-m-p}).$$

The present value of an annuity to begin m years hence and to continue for ever is obtained by making p infinite;

$$\therefore \text{ the present value} = \frac{AR^{-m}}{r}.$$

Example 1. Find the annuity for 10 years which can be purchased for £2000 if the rate of interest be 4 %.

Let the annuity be A pounds.

Then 2000 = the present value of the 1st instalment
+ ...... 2nd ....
+ etc.

$$= \frac{A}{R} + \frac{A}{R^2} + \ldots + \frac{A}{R^{10}}$$

$$= \frac{A(1 - R^{-10})}{r} = \frac{100A}{4}(1 - R^{-10}).$$

$$\therefore A = 80 \div (1 - 1{\cdot}04^{-10}).$$

$$1{\cdot}04^{-10} = \text{antilog}(-10 \log 1{\cdot}04) = \text{antilog}(-0{\cdot}170)$$

$$= \text{antilog } \bar{1}{\cdot}830 = {\cdot}6761.$$

$$\therefore A = 80 \div {\cdot}3239 = 246{\cdot}99.$$

The amount of the annuity is approximately £247.

The following question may be solved like one involving annuities.

Example 2. A sum of £2400 is borrowed now. Payment is made by annual instalments of £192, beginning at the end of a year. How many instalments are required if the rate of interest is 5 per cent.?

Let there be n instalments.

$$\begin{aligned} 2400 &= \text{the present value of the 1st instalment} \\ &+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{2nd} \ldots\ldots \ldots \\ &+ \text{etc.} \\ &= \frac{192}{R} + \frac{192}{R^2} + \ldots + \frac{192}{R^n} = 192 \times \frac{1 - 1\cdot05^{-n}}{\cdot05}. \end{aligned}$$

$$1 - 1\cdot05^{-n} = \frac{2400 \times \cdot05}{192} = \frac{120}{192} = \frac{5}{8}.$$

$$\therefore\ 1\cdot05^{-n} = 1 - \frac{5}{8} = \frac{3}{8}.$$

$$\therefore\ 1\cdot05^{n} = \frac{8}{3}.$$

$$n \log 1\cdot05 = \log 8 - \log 3 = \cdot9031 - \cdot4771 = \cdot4260.$$

$$\therefore\ n = \frac{\cdot4260}{\cdot0212} = 20 \text{ approximately.}$$

Examples. LI. b.

1. What is the present value of a perpetual annuity of £560 at 4 per cent.?

2. What is the present value of an annuity of £425 to continue for 48 years at 5 per cent.?

3. An estate is considered to be worth 25 years' purchase. What is the rate per cent.?

4. A perpetual annuity of £110 is bought for £2000; what is the rate of interest? How many years' purchase is it said to be worth?

5. Find the present value at $3\frac{1}{2}$ per cent. of an annuity of £100 to begin 8 years hence and continue for 15 years. Find also its present value if it is to be perpetual.

6. A person buys an estate yielding £300 a year for £6250. What rate per cent. is he obtaining for his money?

7. If I pay £2000 for the *reversion* of a property after 10 years [*i.e.* for a property which becomes mine after 10 years], what must be the rent from it in order that I may get 5 per cent. on my purchase money?

8. How many years' purchase is a freehold estate worth (i) at $4\frac{1}{2}$ per cent., (ii) at 5 per cent.?

9. Find the present value of an annuity of £75 to begin 4 years hence and continue for 11 years, the rate being 5 per cent.

10. Find the present value of an annuity of £75 to begin now and last for 25 years at 5 per cent.

11. Find the cost of an annuity of £120 to run for 20 years at 5 per cent.

12. Find the present worth of a perpetual annuity of £441 to begin 2 years hence at 5 per cent.

13. I borrow a sum of money, and the debt is to be discharged by monthly payments of £10 for a year. If the rate of interest is 6 per cent. *per annum*, what do I borrow?

14. Find the present value of an annuity of £300 payable every year for 3 years at 5 per cent., the first payment being at the end of a year from now.

15. What should be paid now for the reversion after 8 years of a freehold estate bringing in £2400 a year, if interest be at the rate of 4 per cent.?

16. £3000 borrowed money has to be paid off in 30 years by equal annual instalments. What must each instalment be, if the rate of interest be 4 %?

CHAPTER LII.

THE EXPONENTIAL THEOREM AND LOGARITHMIC SERIES.

348. By the Binomial Theorem, if n be numerically greater than 1,

$$\left(1+\frac{1}{n}\right)^{nx} = 1 + nx\cdot\frac{1}{n} + \frac{nx(nx-1)}{\lfloor\underline{2}}\cdot\frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{\lfloor\underline{3}}\cdot\frac{1}{n^3}+\dots$$

$$= 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\lfloor\underline{2}} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor\underline{3}}+\dots.$$

Also this result is true when $x=1$.

$$\therefore\ \left(1+\frac{1}{n}\right)^{n} = 1 + 1 + \frac{1-\frac{1}{n}}{\lfloor\underline{2}} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor\underline{3}}+\dots;$$

$$\therefore \left\{1 + 1 + \frac{1-\frac{1}{n}}{\lfloor\underline{2}} + \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor\underline{3}}+\dots\right\}^{x} = \left\{\left(1+\frac{1}{n}\right)^{n}\right\}^{x}$$

$$= \left(1+\frac{1}{n}\right)^{nx} = 1 + x + \frac{x\left(x-\frac{1}{n}\right)}{\lfloor\underline{2}} + \frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor\underline{3}}+\dots.$$

This is true for all values of n greater than 1;

$\therefore$ by making n infinite we obtain the result

$$\left(1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\dots\right)^x=1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots.$$

The infinite series $1+\frac{1}{\lfloor 1}+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\dots$ is denoted by e;

$$\therefore\ e^x=1+\frac{x}{\lfloor 1}+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\dots.$$

This is the **Exponential Theorem.**

COR. Let $e^y=a^x$. Then, by taking logs. of both sides to the base e we have $y=x\log_e a$.

$$\therefore\ a^x=e^{x\log_e a}=1+\frac{x\log_e a}{\lfloor 1}+\frac{x^2(\log_e a)^2}{\lfloor 2}+\dots$$

349. The proof of the Exponential Theorem given in Art. 348 is incomplete in some points. The following is preferable.

Let $$f(m)=1+\frac{m}{\lfloor 1}+\frac{m^2}{\lfloor 2}+\dots+\frac{m^r}{\lfloor r}+\dots \qquad (1)$$

then $$f(n)=1+\frac{n}{\lfloor 1}+\frac{n^2}{\lfloor 2}+\dots+\frac{n^r}{\lfloor r}+\dots \qquad (2)$$

Both series on the right are convergent for all finite values of m and n; $\therefore$ the product $f(m)\times f(n)$ may be found by multiplying together the right-hand sides of (1) and (2).

Now in the product of the two series on the right, the terms of the rth degree

$$=\frac{m^r}{\lfloor r}+\frac{m^{r-1}n}{\lfloor r-1\,\lfloor 1}+\frac{m^{r-2}n^2}{\lfloor r-2\,\lfloor 2}+\dots+\frac{n^r}{\lfloor r}$$

$$=\frac{1}{\lfloor r}\left\{m^r+rm^{r-1}n+\frac{r\,.\,\overline{r-1}}{\lfloor 2}m^{r-2}n^2+\dots+n^r\right\}$$

$$=\frac{(m+n)^r}{\lfloor r}=\text{the term of } r\text{th degree in } f(m+n).$$

A similar relation holds for terms of any other degree.

$\therefore$ for all values of m and n we have

$$f(m)\times f(n)=f(m+n). \qquad \dots(3)$$

Hence $f(m).f(n).f(p)=f(m+n).f(p)=f(m+n+p)\ldots$ from (3); and so on.

$$\therefore\ f(m).f(n).f(p)\ldots=f(m+n+p+\ldots).\ \ldots\ldots\ldots\ldots(4)$$

From (4),

$$f(1).f(1).f(1)\ldots \text{ to } x \text{ factors}=f(1+1+1+\ldots \text{ to } x \text{ terms}),$$

$$\textit{i.e.}\quad \{f(1)\}^x=f(x),$$

But

$$\{f(1)\}^x=e^x.$$

$$\therefore\ e^x=f(x)=1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\ldots\ \ldots\ldots\quad \ldots\ldots(5)$$

This proves the theorem for any positive integral index.

Next, to prove it for a positive fractional index, let $x=\frac{h}{k}$, where h and k are positive integers.

Then $\{f(x)\}^k=\left\{f\left(\frac{h}{k}\right)\right\}^k=f\left(\frac{h}{k}\right).f\left(\frac{h}{k}\right)f\left(\frac{h}{k}\right)\ldots$ to k factors

$$=f\left(\frac{h}{k}+\frac{h}{k}+\frac{h}{k}+\ldots \text{ to } k \text{ terms}\right)$$

$$=f(h)\quad e^h.$$

By taking the kth root of both sides we get

$$f(x)-e^{\frac{h}{k}}=e^x.$$

$$\therefore\ e^x=1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\ldots.$$

Thirdly, when x is negative, let $x=-y$. Thus y is positive. From (3) we see that $f(-y).f(y)=f(-y+y)=f(0)=1$.

$$\therefore\ f(-y)=\frac{1}{f(y)}=\frac{1}{e^y}=e^{-y},$$

i.e. $e^x=f(x)=1+\frac{x}{\underline{|1}}+\frac{x^2}{\underline{|2}}+\ldots$ for negative values of x.

350. *The series for e^x is convergent for any finite value of x.*

If we denote the nth term by u_n,

$$u_{n+1}/u_n=\frac{x^{n+1}}{\underline{|n+1}}\cdot\frac{\underline{|n}}{x^n}=\frac{x}{n}.$$

Now this ratio <1 for all terms after the first term for which n becomes numerically greater than x;

$$\therefore\ \text{the series is convergent.}$$

351. The series denoted by e is convergent, and its value is easily calculated to a moderate number of decimal places.

The calculation is simple, because the $(n+1)$th term is found by dividing the nth term by n.

The 1st	term		$=1.$
,, 2nd	,,		$=1.$
,, 3rd	,,	= the 2nd $\div 2$	$=0{\cdot}5.$
,, 4th	,,	= ,, 3rd $\div 3$	$=0{\cdot}166,666,6\ldots.$
,, 5th	,,	= ,, 4th $\div 4$	$=0{\cdot}041,666,6\ldots.$
,, 6th	,,		$=0{\cdot}008,333,3\ldots.$
,, 7th	,,		$=0{\cdot}001,388,8\ldots.$
,, 8th	,,		$=0{\cdot}000,198,4\ldots.$
,, 9th	,,		$=0{\cdot}000,024,8\ldots.$
,, 10th	,,		$=0{\cdot}000,002,7\ldots.$
,, 11th	,,		$=0{\cdot}000,000,2\ldots.$

$\therefore$ by addition, $e = 2{\cdot}71828\ldots.$

The quantity e is very important. With this as base we can calculate by means of series the logarithms of numbers as shown in Art. 352, and these can be converted into *common* logarithms.

352. *Logarithmic Series.*

$$a^x = 1 + x\log_e a + \frac{x^2(\log_e a)^2}{\underline{|2}} + \ldots;$$

$$\therefore\ (1+y)^x = 1 + x\log_e(1+y) + \frac{x^2(\log_e \overline{1+y})^2}{\underline{|2}} + \ldots.$$

But by the Binomial Theorem (**if y be numerically less than 1**),

$$(1+y)^x = 1 + xy + \frac{x(x-1)}{1\,.\,2}y^2 + \frac{x(x-1)(x-2)}{1\,.\,2\,.\,3}y^3 + \ldots.$$

Comparing these two series for $(1+y)^x$, we get, by equating the coefficients of x,

$$\log_e(1+y) = y - \frac{1\,.\,y^2}{1\,.\,2} + \frac{1\,.\,2\,.\,y^3}{1\,.\,2\,.\,3} - \ldots$$

$$= y - \frac{y^2}{2} + \frac{y^3}{3} - \ldots \text{ (if } y < 1 \text{ numerically).} \quad \ldots(1)$$

Similarly (or by writing $-y$ for y),

$$\log_e(1-y) = -y - \frac{y^2}{2} - \frac{y^3}{3} - \dots, \qquad \dots\dots\dots\dots\dots(2)$$

and from these, by subtraction,

$$\log_e \frac{1+y}{1-y} = 2\left\{y + \frac{y^3}{3} + \frac{y^5}{5} + \dots\right\}. \qquad \dots\dots\dots\dots\dots(3)$$

We can convert $\dfrac{1+y}{1-y}$ into $\dfrac{n+1}{n}$ by putting $\dfrac{1}{2n+1}$ for y.

Thus $\quad \log_e(n+1) - \log_e n = \log_e \dfrac{n+1}{n} = \log_e \dfrac{1+\dfrac{1}{2n+1}}{1-\dfrac{1}{2n+1}}$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \dots\right\}, \text{ from (3).}$$

This series enables us to calculate $\log_e 2$ by putting $n=1$, $\log_e 3$ from $\log_e 2$ by putting $n=2$; and so on.

Thus the logarithms of all numbers to base e can be calculated.

By Art. 247 we know that all these can be converted into *common* logarithms by multiplying by $\log_{10} e$, *i.e.* by $\dfrac{1}{\log_e 10}$, *i.e.* by $\cdot 43429\dots$.

This multiplier is known as the *modulus*.

353. *Calculation of* $\log_e 2$.

As in the last article, $\quad \log(n+1) - \log(n)$

$$= \log\frac{n+1}{n} = \log\frac{1+\dfrac{1}{2n+1}}{1-\dfrac{1}{2n+1}}$$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \dots\right\}.$$

By putting $n=1$, since $\log_e 1 = 0$, we get

$$\log_e 2 = 2\left\{\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3^3} + \frac{1}{5}\cdot\frac{1}{3^5} + \dots\right\}.$$

Now

$$\begin{array}{llll}
\frac{1}{3} = \cdot 333,333,3 \ldots, & & \frac{1}{3} = \cdot 333,333,3 \ldots, \\
\frac{1}{3^3} = \cdot 037,037,0 \ldots, & & \frac{1}{3} \cdot \frac{1}{3^3} = \cdot 012,345,6 \ldots, \\
\frac{1}{3^5} = \cdot 004,115,2 \ldots, & & \frac{1}{5} \cdot \frac{1}{3^5} = \cdot 000,823,0 \ldots, \\
\frac{1}{3^7} = \cdot 000,457,2 \ldots, & & \frac{1}{7} \cdot \frac{1}{3^7} = \cdot 000,065,3 \ldots, \\
\frac{1}{3^9} = \cdot 000,050,8 \ldots, & & \frac{1}{9} \cdot \frac{1}{3^9} = \cdot 000,005,6 \ldots, \\
\frac{1}{3^{11}} = \cdot 000,005,6 \ldots, & & &
\end{array}$$

$$\therefore \text{ by addition, } \tfrac{1}{2}\log 2 = \cdot 346573 \ldots$$

Now $\frac{1}{11} \cdot \frac{1}{3^{11}}$ begins with 6 zeros, $\qquad \therefore$

from this value of $\frac{1}{2}\log_e 2$ we get $\log_e 2$ correct certainly to 4 decimal places.

Thus $\qquad \log_e 2 = \cdot 6931.$

Example 1. Simplify $1 - \frac{1}{3|\underline{1}} + \frac{1}{9|\underline{2}} - \frac{1}{27|\underline{3}} + \frac{1}{81|\underline{4}} - \text{etc.}$

This is the expansion of e^x with $-\frac{1}{3}$ for x;

$$\therefore \text{ the series} = e^{-\frac{1}{3}}.$$

Example 2. Expand $\log(1+7x+12x^2)$ in ascending powers of x, if x is numerically less than $\frac{1}{4}$.

$$\log(1+7x+12x^2) = \log\{(1+3x)(1+4x)\} = \log(1+3x) + \log(1+4x).$$

These two logarithms can be expanded in ascending powers of x since numerically $4x < 1$, and $\therefore 3x < 1$.

Example 3. Sum to infinity the series whose nth term is $\frac{n^3}{|\underline{n}}$.

$$\text{The } n\text{th term} = \frac{n^3}{|\underline{n}} = \frac{n^2}{|\underline{n-1}} = \frac{1}{|\underline{n-1}} + \frac{n^2-1}{|\underline{n-1}}$$

$$= \frac{1}{|\underline{n-1}} + \frac{n+1}{|\underline{n-2}} = \frac{1}{|\underline{n-1}} + \frac{3}{|\underline{n-2}} + \frac{n-2}{|\underline{n-2}}$$

$$= \frac{1}{|\underline{n-1}} + \frac{3}{|\underline{n-2}} + \frac{1}{|\underline{n-3}}.$$

$$\text{The } (n-1)\text{th term} = \frac{1}{|\underline{n-2}} + \frac{3}{|\underline{n-3}} + \frac{1}{|\underline{n-4}},$$

The 4th term $=\dfrac{1}{\underline{|3}}+\dfrac{3}{\underline{|2}}+\dfrac{1}{\underline{|1}}$.

The 3rd term $=\dfrac{1}{\underline{|2}}+\dfrac{3}{\underline{|1}}+1$.

The 2nd term $=\dfrac{1}{\underline{|1}}+3$.

The 1st term $=1$.

$\therefore$ by adding up the columns and making n infinite, we get

$$\text{the series}=e+3e+e=5e.$$

NOTE. As in numerical examples of logarithms it is assumed that 10 is the base when no base is mentioned, so in the logarithm of an algebraic literal quantity the base is assumed to be e if none is named.

Examples. LII.

1. Prove that $\left\{1+\dfrac{1}{\underline{|1}}+\dfrac{1}{\underline{|2}}+\dfrac{1}{\underline{|3}}+\dots\right\}\left\{1-\dfrac{1}{\underline{|1}}+\dfrac{1}{\underline{|2}}-\dfrac{1}{\underline{|3}}+\dots\right\}=1$.

2. Find the value of $1-\dfrac{2}{\underline{|1}}+\dfrac{2^2}{\underline{|2}}-\dfrac{2^3}{\underline{|3}}+\dots$ to infinity.

3. Find the value of $3^{-1}-\frac{1}{2}.3^{-2}+\frac{1}{3}.3^{-3}-\frac{1}{4}.3^{-4}+\dots$ to infinity.

4. Prove that $\dfrac{x-y}{x}+\dfrac{(x-y)^2}{2x^2}+\dfrac{(x-y)^3}{3x^3}+\dots$ to infinity $=\log\dfrac{x}{y}$, if $\dfrac{y}{x}$ is a positive proper fraction.

5. Prove that $\dfrac{1}{e}=\dfrac{2}{\underline{|3}}+\dfrac{4}{\underline{|5}}+\dfrac{6}{\underline{|7}}+\dots$.

6. Expand $\log(1+3x+2x^2)$ in ascending powers of x, where x is numerically less than $\frac{1}{2}$.

7. Expand $\log\dfrac{1-2x}{1-3x}$, where x is numerically less than $\frac{1}{3}$.

8. Find $\log_e 1{\cdot}04$ to 4 decimal places by the series for $\log(1+x)$.

9. Expand $\log(1-x+x^2)$ in powers of x, where x is numerically less than 1.

10. Sum the series $x^2+\dfrac{1}{y^2}-\dfrac{1}{2}\left(x^4+\dfrac{1}{y^4}\right)+\dfrac{1}{3}\left(x^6+\dfrac{1}{y^6}\right)\dots$, x being <1 and $y>1$ numerically.

11. Given $\log_e 2={\cdot}693147$, find $\log_e 3$ from the series for $\log_e(n+1)-\log_e n$.

12. Given $\log_e 3=1{\cdot}098612$, calculate $\log_e 10$ by a series, and hence find $\log_{10} e$ to 5 decimal places.

13. Sum the series $\dfrac{1}{\underline{|2}}+\dfrac{2}{\underline{|3}}+\dfrac{3}{\underline{|4}}+\dots$ *ad inf.*

14. Sum the series $\dfrac{1}{x}+\dfrac{1}{3x^3}+\dfrac{1}{5x^5}+\dots$ *ad inf.* if $x>1$.

15. Sum the series $1^2+\frac{2^2}{\lfloor 2}+\frac{3^2}{\lfloor 3}+\frac{4^2}{\lfloor 4}+\dots$ *ad inf.*

16. Prove that, when $x<1$, $\log_e\frac{1+x}{1-x}=2\left\{x+\frac{x^3}{3}+\frac{x^5}{5}+\dots\right\}$, and find $\log_e\frac{2001}{1999}$ to 11 decimal places. $\left[\text{Let } x=\frac{1}{2000}.\right]$

17. Prove that $1+\frac{1+2}{\lfloor 1}+\frac{1+2+3}{\lfloor 2}+\dots$ *ad inf.* $=\frac{7e}{2}$.

18. Prove that $\frac{1}{2}\log_e n=\frac{n-1}{n+1}+\frac{1}{3}\left(\frac{n-1}{n+1}\right)^3+\frac{1}{5}\left(\frac{n-1}{n+1}\right)^5+\dots$.

19. Find the coefficient of x^3 in the expansion of $\frac{1}{\{(1+x)e^x\}}$.

20. Prove that

$$\log_e(n+1)-\log_e n=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\dots\right\}.$$

Hence show that $\log_e 13=2\log_e 2+\log_e 3+\cdot 080043$ correct to 6 decimal places.

21. Show that the difference of the logarithms of consecutive integers continually diminishes.

22. If

$$x=y+\frac{y^2}{\lfloor 2}+\frac{y^3}{\lfloor 3}+\dots \text{ to infinity,}$$

$$\text{then } y=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\dots \text{ to infinity.}$$

23. If α, β are the roots of the equation $x^2+px+q=0$,

$$\log(1-px+qx^2)=(\alpha+\beta)x-\frac{\alpha^2+\beta^2}{2}x^2+\frac{\alpha^3+\beta^3}{3}x^3-\dots,$$

if αx and βx are numerically <1.

24. Prove that $\log_e(n+1)+\log_e(n-1)-2\log_e n$

$$=-2\left\{\frac{1}{2n^2-1}+\frac{1}{3(2n^2-1)^3}+\frac{1}{5(2n^2-1)^5}+\dots\right\}.$$

25. Prove that $\dfrac{a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\dots}{b-1-\frac{1}{2}(b-1)^2+\frac{1}{3}(b-1)^3-\dots}=\log_b a$,

if a and b are numerically <2.

26. Prove that the coefficient of x^n in the expansion of

$$1+\frac{a+x}{\lfloor 1}+\frac{(a+x)^2}{\lfloor 2}+\frac{(a+x)^3}{\lfloor 3}+\dots \text{ is } \frac{e^a}{\lfloor n}.$$

27. Prove that $\frac{1}{1.2}-\frac{1}{2.3}+\frac{1}{3.4}-\frac{1}{4.5}+\dots$ *ad inf.* $=2\log_e 2-1$.

28. If x is numerically less than 1,

$$\frac{x}{1+x^2}+\frac{1}{3}\left(\frac{x}{1+x^2}\right)^3+\frac{1}{5}\left(\frac{x}{1+x^2}\right)^5+\dots=x-\frac{2x^3}{3}+\frac{x^5}{5}+\frac{x^7}{7}-\frac{2x^9}{9}+\dots$$

29. Prove that $\log_e 3=1+\frac{1}{3.2^2}+\frac{1}{5.2^4}+\frac{1}{7.2^6}+\dots$ *ad inf.*

CHAPTER LIII.

INDETERMINATE OR ARBITRARY COEFFICIENTS.

354. The use of indeterminate coefficients will be best explained by means of examples.

Example 1. Find the condition that x^3+qx+r may have a factor of the form $(x+a)^2$.

Since one factor ends with a^2, the other must end with $\frac{r}{a^2}$.

$$\therefore\ x^3+qx+r=(x^2+2ax+a^2)\left(x+\frac{r}{a^2}\right)$$

$$=x^3+\left(2a+\frac{r}{a^2}\right)x^2+\left(\frac{2r}{a}+a^2\right)x+r.$$

These two expressions are identically equal, by hypothesis.

$\therefore$ by equating coefficients, we have $0=2a+\frac{r}{a^2}$, $\quad q=\frac{2r}{a}+a^2$;

$$\therefore\ 2a^3=-r;$$

$$\therefore\quad q=-4a^2+a^2=-3a^2;$$

$$\therefore\ \left(\frac{q}{-3}\right)^3=\left(\frac{r}{-2}\right)^2;$$

$$\therefore\ 4q^3+27r^2=0.$$

Example 2. Find values (different from a and b) for A and B, which will render the equation $m(x+\mathsf{A})^2+n(x+\mathsf{B})^2\equiv m(x+a)^2+n(x+b)^2$ an identity for all values of x.

Multiplying out,

$$2mx\mathsf{A}+m\mathsf{A}^2+2nx\mathsf{B}+n\mathsf{B}^2\equiv 2mxa+ma^2+2nxb+nb^2.$$

Since this is an identity, the coefficients of x on either side are equal, and the constant terms are equal.

$$\therefore\ m\mathsf{A}+n\mathsf{B}=ma+nb, \quad\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and $$m\mathsf{A}^2+n\mathsf{B}^2=ma^2+nb^2.$$

These may be written $\quad m(\mathsf{A}-a)=-n(\mathsf{B}-b),$

$$m(\mathsf{A}^2-a^2)=-n(\mathsf{B}^2-b^2).$$

Dividing and neglecting the solution $\mathsf{A}=a$, $\mathsf{B}=b$, we find that

$$\mathsf{A}+a=\mathsf{B}+b.$$

Multiplying this by m and subtracting from (1),

$$\mathsf{B}=\frac{2am+bn-bm}{m+n},$$

and hence $$\mathsf{A}=\frac{2bn+am-an}{m+n}.$$

Example 3. Find the values of a, b, and c when

$$5x^2 - 3x - 10 = a(x-1)(x-2) + b(x-2)(x+3) + c(x+3)(x-1).$$

The equation is true for all values of x.

$$\therefore \text{ putting } x = 1, \text{ we have } -8 = -4b; \quad \therefore b = 2;$$
$$x = 2, \quad 4 = 5c; \quad \therefore c = \tfrac{4}{5};$$
$$x = -3, \quad 44 = 20a; \quad \therefore a = \tfrac{11}{5}.$$

355. *If $a + bx + cx^2 + \ldots = a_1 + b_1x + c_1x^2 + \ldots$ for all values of x for which the two series are convergent, then $a = a_1$, $b = b_1$, $c = c_1$, etc.*

Since the series are convergent, their difference is convergent;

$$\therefore a - a_1 + (b - b_1)x + (c - c_1)x^2 + \ldots = 0 \quad \ldots\ldots\ldots\ldots(1)$$

for all values of x for which the series is convergent.

Zero is such a value; $\therefore$ by putting $x = 0$, we get

$$a - a_1 = 0, \quad i.e. \quad a = a_1.$$

This result is independent of x.

$\therefore$ from (1), we find that, for other values of x besides zero,

$$(b - b_1)x + (c - c_1)x^2 + \ldots = 0;$$
$$\therefore b - b_1 + (c - c_1)x + \ldots = 0;$$

$\therefore$ by the foregoing part, $b = b_1$; and so on.

356. *Expansion of fractions into series.*

The fraction $\dfrac{1}{1-x}$ may be expressed as an infinite series (x being numerically less than 1), (1) by division, (2) by the Binomial Theorem, (3) by indeterminate coefficients.

(1) By division, $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots.$

(2) By the Binomial Theorem,

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots.$$

(3) Let $\dfrac{1}{1-x} = a + bx + cx^2 + dx^3 + \ldots.$

Multiplying both sides by $1 - x$, we get

$$\begin{aligned} 1 &= a + bx + cx^2 + dx^3 + \ldots \\ &\quad\;\; - ax - bx^2 - cx^3 - \ldots \\ &= a + (b-a)x + (c-b)x^2 + (d-c)x^3 + \ldots \end{aligned}$$

Equate coeffts. of like powers of x on the two sides.

Then $$a=1;$$
$$b-a=0; \quad \therefore\ b=a=1.$$
$$c-b=0; \quad \therefore\ c=b=1.$$
$$d-c=0; \quad \therefore\ d=c=1\ldots.$$
$$\therefore\ \frac{1}{1-x}=1+x+x^2+x^3+\ldots.$$

By putting $-x$ for x, we obtain
$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots.$$

Similarly, $$\frac{1}{1-px}=1+px+p^2x^2+p^3x^3+\ldots.$$

357. *A fraction of the form* $\dfrac{p+qx}{1+mx+nx^2}$ *may be expanded by indeterminate coefficients.*

Let $$\frac{p+qx}{1+mx+nx^2}=a+bx+cx^2+dx^3+\ldots.$$
$$\therefore\ p+qx=(1+mx+nx^2)(a+bx+cx^2+dx^3+\ldots)$$
$$=a+(b+ma)x+(c+mb+na)x^2+(d+mc+nb)x^3+\ldots.$$

By equating coefficients of like powers of x, we get
$$a=p, \qquad b+ma=q,$$
$$c+mb+na=0, \quad d+mc+nd=0, \text{ etc.}$$

From these equations we determine successively a, b, c, etc., and so obtain the required expansion.

Example 1. Expand $\dfrac{1+2x-3x^2}{1-2x+3x^2}$ in ascending powers of x.

Let $$\frac{1+2x-3x^2}{1-2x+3x^2}=a+bx+cx^2+dx^3+ex^4+\ldots.$$

Then $1+2x-3x^2=(1-2x+3x^2)(a+bx+cx^2+\ldots)$
$$=a+bx+cx^2+dx^3+ex^4+\ldots$$
$$-2ax-2bx^2-2cx^3-2dx^4-\ldots$$
$$+3ax^2+3bx^3+3cx^4+\ldots$$
$$=a+(b-2a)x+(c-2b+3a)x^2+(d-2c+3b)x^3+(e-2d+3c)x^4+\ldots.$$

Equate coefficients.
$$a=1.$$
$$b-2a=2; \quad \therefore\ b=2a+2=4.$$
$$c-2b+3a=-3; \quad \therefore\ c=2b-3a-3=8-3-3=2.$$
$$d-2c+3b=0; \quad \therefore\ d=2c-3b=4-12=-8.$$
$$e-2d+3c=0; \quad \therefore\ e=-16-6=-22, \text{ etc.}$$
$$\therefore\ \text{the fraction}=1+4x+2x^2-8x^3-22x^4-\ldots.$$

Example 2. The reverse process may be employed.

Find the fraction from which may be obtained the series

$$1+4x+2x^2-8x^3-22x^4-\dots,$$

assuming that some relation holds between every three consecutive coefficients.

Let $$\frac{a+bx+cx^2}{1+px+qx^2}=1+4x+2x^2-8x^3-22x^4\dots.$$

Then, by multiplication,

$$\begin{aligned} a+bx+cx^2 = 1+4x+2x^2-8x^3-22x^4\dots \\ +px+4px^2+2px^3-8px^4\dots \\ +qx^2+4qx^3+2qx^4\dots. \end{aligned}$$

Equate coefficients.

Then
$$a=1, \quad\dots\dots(1)$$
$$b=4+p, \quad\dots\dots(2)$$
$$c=2+4p+q, \quad\dots\dots(3)$$
$$0=-8+2p+4q, \quad\dots\dots(4)$$
$$0=-22-8p+2q. \quad\dots\dots(5)$$

From (4) and (5), $2p+4q=8$, *i.e.* $p+2q=4$,

and $-8p+2q=22$.

From these simultaneous equations,

$$9p=-18;$$
$$\therefore\ p=-2;$$
$$\therefore\ q=3.$$

From (2), $b=4-2=2$.

From (3), $c=2-8+3=-3$.

From (1), $a=1$.

Substituting these values in $\dfrac{a+bx+cx^2}{1+px+qx^2}$, we obtain the required fraction $\dfrac{1+2x-3x^2}{1-2x+3x^2}$.

Examples. LIII. a.

1. If $(x+a)(2x^2-bx+3)\equiv 2x^3+5x^2+5x+6$, find the values of a and b.

2. If $(x^2-ax-1)(x^2+4x-3b)\equiv x^4-16x^2-8x-1$, find the values of a and b.

3. If $x^4+x^2+1\equiv(x^2+ax+2b)(x^2+cx+1)$, find the values of a, b, c.

Evaluate a, b, c:

4. When $3x^2-5x+4\equiv a(x-2)(x-3)+b(x-3)(x-1)+c(x-1)(x-2)$.

5. When $2x^2+11x-11\equiv a(x+2)(x-3)+b(x-3)(x+1)+c(x+1)(x+2)$.

Evaluate A, B, C, D:

6. When $3x+2\equiv \mathrm{A}(x+1)^3+\mathrm{B}x(x+1)^2+\mathrm{C}x(x+1)+\mathrm{D}x$.

7. When $x^3-x^2+3x-2\equiv \mathrm{A}+\mathrm{B}(x-1)+\mathrm{C}(x-1)(x-2)$ $+\mathrm{D}(x-1)(x-2)(x-3)$.

8. Find the value of k which makes x^3+7x^2+8x+k exactly divisible by $x+3$.

9. If $2(x+c)(x+2d)+2(x+2c)(x+d)$ be an exact square, then
$$9c^2-14cd+9d^2=0.$$

10. Express $2x^3-4x^2+5$ as an integral function of $x-1$.

11. ,, $5x^3-8x^2+4x+1$,, ,, ,,

12. ,, $3x^3-x^2-2x+1$,, ,, $x-2$.

13. ,, x^3+4x^2-x+7 ,, ,, $x+1$.

Find values of a and b which make

14. x^3+ax^2+x+b exactly divisible by x^2+2x+3.

15. $4x^7+3x^6+ax^2+bx+1$,, ,, x^2-x+1.

If each of the following expressions resolve into linear factors, find the value of k in each case:

16. $x^2-3xy+2y^2+x+2y+k$. 17. $6x^2-13xy-8y^2-6x-22y+k$.

18. $kx^2-y^2-4x+8y-15$.

19. Find the relation between p, q, r, that $x^4+2ax^3+px^2+qx+r$ may be a square.

20. Find the condition that x^3+px^2+qx+r may contain a factor of the form x^2-a^2.

21. Find a and b if x^3+ax^2+bx+1 is divisible by $(x+1)^2$.

22. Find a and b if $2x^3+3x^2+ax+b$ is divisible by $(x-1)(x-2)$.

23. Find c so that $3x^3+4x^2+cx+4$ may be divisible by x^2+1.

24. If $a+b+c=0$, $ax^5+bx^4+cx^3+cx^2+bx+a$ is divisible by x^2-1.

25. If $3x^2+8x+11$ can be put in the form
$$a(x+1)(x+2)+b(x+2)(x+3)+c(x+3)(x+1), \text{ find } a, b, c.$$

26. Express $5x-6$ in the form
$$a(x-2)(x-3)+b(x-3)(x-1)+c(x-1)(x-2).$$

27. Express $x^4+5x^3+8x^2+2x+1$ in the form
$$a+b(x+2)+c(x+2)^2+d(x+2)^3+e(x+2)^4.$$

[On dividing by $x+2$, and then the quotient by $x+2$, and so on, we shall find that the successive remainders give the values of a, b, c ....]

Verify the result of the last question by putting $y-2$ for x in $x^4+5x^3+8x^2+2x+1$ and expanding in powers of y, *i.e.* in powers of $x+2$.

By indeterminate coefficients expand the following into series in ascending powers of x:

28. $\dfrac{1}{2+3x}$. 29. $\dfrac{1-x}{1-2x-2x^2}$. 30. $\dfrac{1-2x}{1-4x+x^2}$. 31. $\dfrac{1-3x}{1-6x+8x^2}$.

Find the fractions which will expand into the following series:

32. $1-x^2-x^3+x^5+x^6-\ldots$. 33. $1+2x+3x^2+4x^3+\ldots$.

34. $2+5x+17x^2+65x^3+\ldots$. 35. $1+3x+4x^2+7x^3+11x^4+\ldots$.

PARTIAL FRACTIONS.

358. The sum of two or more fractions can be expressed as a single fraction. It is often convenient to use the reverse process, and resolve a single fraction into the sum of simpler fractions, **partial** fractions as they are called.

In an algebraic fraction, if the numerator is of lower degree than the denominator, the fraction is called a **proper fraction**; otherwise it is an **improper** fraction.

359. *Resolve* $\dfrac{4-9x}{(1-2x)(1-3x)}$ *into partial fractions.*

Let
$$\frac{4-9x}{(1-2x)(1-3x)} = \frac{\mathsf{A}}{1-2x}+\frac{\mathsf{B}}{1-3x}. \qquad (1)$$

Here A and B are constants (*i.e.* quantities independent of x), whose values have to be determined.

From (1), $\qquad 4-9x=\mathsf{A}(1-3x)+\mathsf{B}(1-2x).$

The two sides of this equation have to be true for all values of x.

Hence, putting $x=\frac{1}{3}$, we have $4-3=\mathsf{B}(1-\frac{2}{3})$, whence $\mathsf{B}=3$.

Also $\qquad$ „ $\quad x=\frac{1}{2}$, $\quad$ „ $\quad 4-\frac{9}{2}=\mathsf{A}(1-\frac{3}{2})$, whence $\mathsf{A}=1$.

$$\therefore \frac{4-9x}{(1-2x)(1-3x)}=\frac{1}{1-2x}+\frac{3}{1-3x}.$$

COROLLARY. If $3x<1$, $2x$ is also <1.

$\therefore$ by the Binomial Theorem, or by division,

$$\frac{4-3x}{(1-2x)(1-3x)}=(1+2x+2^2x^2+2^3x^3+\dots \textit{ad inf.})$$
$$+3(1+3x+3^2x^2+3^3x^3+\dots \textit{ad inf.}).$$

360. A fraction of the form $\dfrac{ax^{n-1}+bx^{n-2}+cx^{n-3}+\dots+qx+r}{(x-1)^n}$ may be expressed in the form

$$\frac{\mathsf{A}}{x-1}+\frac{\mathsf{B}}{(x-1)^2}+\frac{\mathsf{C}}{(x-1)^3}+\dots+\frac{\mathsf{Q}}{(x-1)^n}.$$

Take as an example the fraction $\dfrac{3x^2-x+4}{(x-1)^3}$.

$$\text{The fraction} = \frac{3(x-1)^2+6x-3-x+4}{(x-1)^3}$$
$$= \frac{3(x-1)^2+5(x-1)+6}{(x-1)^3}$$
$$= \frac{3}{x-1}+\frac{5}{(x-1)^2}+\frac{6}{(x-1)^3}.$$

We might get this by putting $x=y+1$, or by the method of Ex. LIII. a. 27.

361. *Resolve* $\frac{4+5x}{(1-x)^3}$ *into partial fractions.*

Let $$\frac{4+5x}{(1-x)^3} \equiv \frac{A}{1-x}+\frac{B}{(1-x)^2}+\frac{C}{(1-x)^3}.$$

Then $$4+5x \equiv A(1-x)^2+B(1-x)+C.$$

Putting $x=1$, we have $C=9$.

To obtain the values of B and C, we equate coefficients.

Equating coefficients of x^2, $A=0$.

" " x, $5=-2A-B$; $\therefore B=-5$;

$$\therefore \frac{4+5x}{(1-x)^3} \equiv \frac{9}{(1-x)^3}-\frac{5}{(1-x)^2}.$$

This question might be solved thus:

Put $1-x=y$, so that $x=1-y$.

Then $$\frac{4+5x}{(1-x)^3} \equiv \frac{4+5-5y}{y^3}=\frac{9}{y^3}-\frac{5}{y^2}$$
$$=\frac{9}{(1-x)^3}-\frac{5}{(1-x)^2}.$$

362. We will now consider the case in which a factor of the denominator is a quadratic expression which does not contain simple factors; *e.g.* $\frac{3x^2+7x+2}{(x+1)(x^2+2x+5)}$.

Let the fraction be equal to $\frac{A}{x+1}+\frac{Bx+C}{x^2+2x+5}$.

Then $3x^2+7x+2 \equiv A(x^2+2x+5)+(Bx+C)(x+1)$.

Equate coefficients. Then $A+B=3$, $2A+B+C=7$, $5A+C=2$.

$$\therefore A=-\tfrac{1}{2},\ B=\tfrac{7}{2},\ C=\tfrac{9}{2};$$

$$\text{the fraction} = \frac{7x+9}{2(x^2+2x+5)} - \frac{1}{2(x+1)}.$$

If we had taken $\frac{\mathsf{A}}{x+1}$ and $\frac{\mathsf{B}}{x^2+2x+5}$ as the partial fractions, we should have obtained 3 (inconsistent) equations to determine the 2 quantities A and B.

If an improper fraction has to be resolved into partial fractions, it must first be reduced by division till the fractional part is a proper fraction.

For instance, $\frac{x^3+3x^2+x+4}{(x+1)(x+3)}$ must be reduced by division to the form $x-1+\frac{2x+7}{(x+1)(x+3)}$.

363. *Resolve* $\frac{7-3x+4x^2}{(1+x^2)(1-x)}$ *into partial fractions. Hence expand the expression in ascending powers of x (x being < 1), and write down the coefficient of x^n in this expansion.*

Let $$\frac{7-3x+4x^2}{(1+x^2)(1-x)} \equiv \frac{\mathsf{A}x+\mathsf{B}}{1+x^2} + \frac{\mathsf{C}}{1-x}.$$

Then $$7-3x+4x^2 \equiv (\mathsf{A}x+\mathsf{B})(1-x)+\mathsf{C}(1+x^2).$$

Putting $x=1$, we have $8=2\mathsf{C}$; $\therefore\ \mathsf{C}=4$.

Equating coefficients of x^2,

$$4=-\mathsf{A}+\mathsf{C}=-\mathsf{A}+4;\quad \therefore\ \mathsf{A}=0.$$

Equating the constant terms,

$$7=\mathsf{B}+\mathsf{C}=\mathsf{B}+4;\quad \therefore\ \mathsf{B}=3.$$

$$\therefore\ \text{the expression} = \frac{3}{1+x^2}+\frac{4}{1-x}$$
$$=3(1+x^2)^{-1}+4(1-x)^{-1}.$$

We may use the Binomial Theorem here, for $x<1$.

$$\therefore\ \text{the expression} = 3[1-x^2+x^4-x^6+\ldots+(-x^2)^r+\ldots] \quad \ldots(1)$$
$$+4[1+x+x^2+x^3+\ldots+x^p+\ldots] \quad \ldots\ldots(2)$$
$$=7+4x+x^2+4x^3+7x^4+4x^5\ldots.$$

If n is *even*, the coefficient of x^n is $3(-1)^{\frac{n}{2}}+4$;

" *odd*, " " 4, from lines (1) and (2).

Examples. LIII b.

Resolve into partial fractions

1. $\dfrac{2x}{x^2-1}$. 2. $\dfrac{x-4}{(x-2)(x-3)}$. 3. $\dfrac{4}{x^2-4}$.

4. $\dfrac{1}{x^2+4x+3}$. 5. $\dfrac{2x-1}{(1-x)(2-x)}$. 6. $\dfrac{1}{x^2-x-2}$.

7. $\dfrac{1}{(x-a)(x-b)}$. 8. $\dfrac{x-10}{(x-1)(x+2)}$. 9. $\dfrac{x-2}{6x^2-7x+2}$.

10. $\dfrac{2}{(x+1)(x+2)(x+3)}$. 11. $\dfrac{7x+5}{3x^2+4x+1}$. 12. $\dfrac{2x-3}{x(4x^2-1)}$.

13. $\dfrac{6x^2-4x-6}{(x-1)(x-2)(x-3)}$. 14. $\dfrac{3x^2+6x-29}{(x-3)(x^2-1)}$. 15. $\dfrac{x}{x^3-1}$.

16. $\dfrac{1}{(x-1)^2(x^2+1)}$. 17. $\dfrac{5x^2-4x+6}{2(x-2)(x+1)^2}$. 18. $\dfrac{x^3-3x}{(x^2-x+1)^2}$.

19. $\dfrac{x^3}{(x-1)^4}$. 20. $\dfrac{4(x+10)}{x^4-16}$. 21. $\dfrac{1}{x^2(x^2-1)}$.

22. $\dfrac{16}{(x-2)(x+2)^2}$. 23. $\dfrac{x^2}{(x^2-1)(x-2)}$. 24. $\dfrac{x^3+x^2-x+3}{(x-1)(x+2)}$.

25. $\dfrac{x-1}{(x+2)(x+3)^2}$. 26. $\dfrac{25}{(x+2)^2(x^2+1)}$. 27. $\dfrac{x^3+4x^2+6x+1}{x(x^2+x+1)}$.

28. $\dfrac{12x^2-30x+19}{(2x-1)(2x-3)(3x-5)}$. 29. $\dfrac{1-3x+2x^2-x^3}{(x-1)^2(x^2-x+1)}$. 30. $\dfrac{8-8x}{x(x-2)}$.

Resolve into partial fractions, expand in powers of x, assuming that values of x are taken which make the series convergent, and give the general term of

31. $\dfrac{x}{1-5x+6x^2}$. 32. $\dfrac{10x+1}{1-x-12x^2}$.

33. $\dfrac{5}{x^2-3x-4}$. 34. $\dfrac{1}{(1-2x)(1-3x)(1-4x)}$.

35. $\dfrac{x}{(1-ax)(1-bx)(1-cx)}$. By expanding $\dfrac{1}{1-ax}$, etc., on both sides, multiplying together those on the left and equating coefficients, find the factors of

$$a^2(b-c)+b^2(c-a)+c^2(a-b) \text{ and of } a^3(b-c)+b^3(c-a)+c^3(a-b).$$

36. $\dfrac{7(3x-1)}{(2-3x)(1+2x)}$. 37. $\dfrac{4x^3}{x^2-3x+2}$. 38. $\dfrac{2x+3}{x^2+x-2}$.

39. Resolve into partial fractions $\dfrac{3x-4}{(x-2)(x-1)^3}$.

40. $\dfrac{x^2+1}{x^3-1}$.

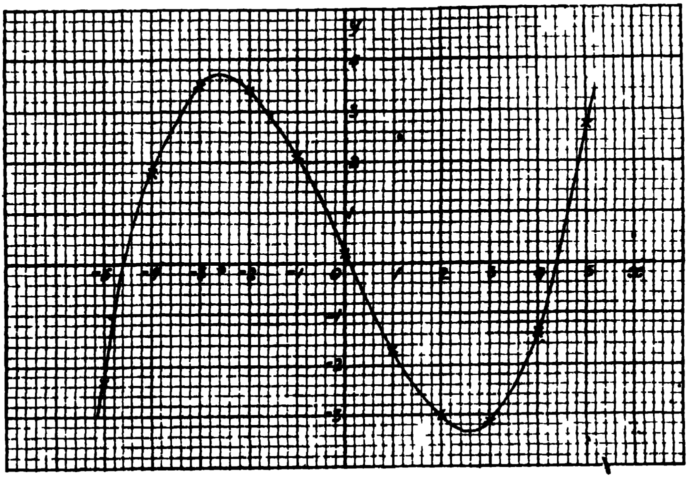
41. The coefficient of x^n in the expansion of $\dfrac{x}{(1-x)(1-cx)}$ is $\dfrac{1-c^n}{1-c}$.

CHAPTER LIV.

MISCELLANEOUS CURVES SOLUTION OF CUBIC EQUATIONS.

364. *Draw the graph of the equation* $y = \frac{x^3 - 20x + 2}{10}$, *between the points given by* $x = -5$ *and* $x = 5$.

When	$x =$	-5	-4	-3	-2	-1	0	1	2	3	4	5
	$x^3 =$	-125	-64	-27	-8	-1	0	1	8	27	64	125
	x^3+2	-123	-62	-25	-6	1	2	3	10	29	66	127
	$20x$	-100	-80	-60	-40	-20	0	20	40	60	80	100
$\therefore$	$10y =$	-23	18	35	34	21	2	-17	-30	-31	-14	27
$\therefore$	$y =$	-2.3	1.8	3.5	3.4	2.1	$.2$	-1.7	-3	-3.1	-1.4	2.7



Plotting these points and joining them by an even curve, we have the graph as shown in the diagram. [The printed figure is reduced in size].

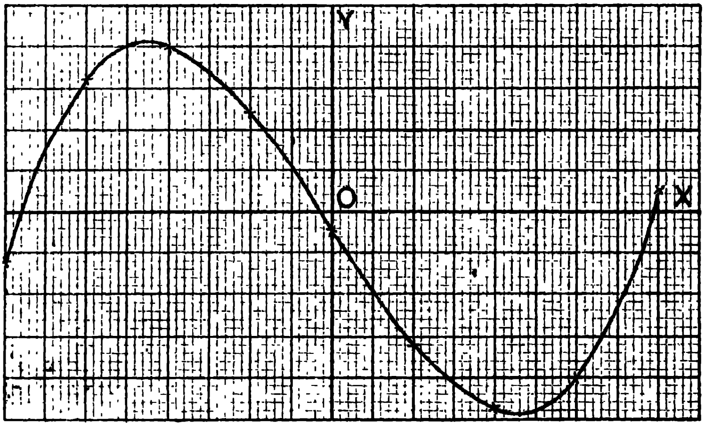
COROLLARY. From the graph we see that the greatest value of the expression $\frac{x^3 - 20x + 2}{10}$ is 3·64, and its least value is $-3·24$, excluding infinite values.

It will be found that the graphs of all equations of the form $y = ax^3 + bx^2 + cx + d$ are in the shape of the letter S.

365. *Solve the equation* $x^3 - 15x - 2 = 0$.

Let $y = x^3 - 15x - 2$.

When	$x =$	-4	-3	-2	-1	0				
	$x^3 =$	-64		-8	-1	0			27	64
	$x^3 - 2 =$	-66	-29	-10	-3	-2	-1	6	25	62
	$15x =$	-60	-45	-30	-15	0	15	30	45	60
∴	$y =$	-6		20	12	-2	-16	-24	-20	



Use 1 inch as x unit, and one-tenth of an inch as y unit.

∴ Plotting these points and joining them by an even curve, we have the graph as shown in the diagram. [The printed graph is reduced in size.]

We use a large x unit because we wish to find x as accurately as possible.

Where the curve cuts the axis of x,

$$x = -3{\cdot}82, \quad -0{\cdot}13, \quad 3{\cdot}95,$$

which are the required roots, for where x has these values,

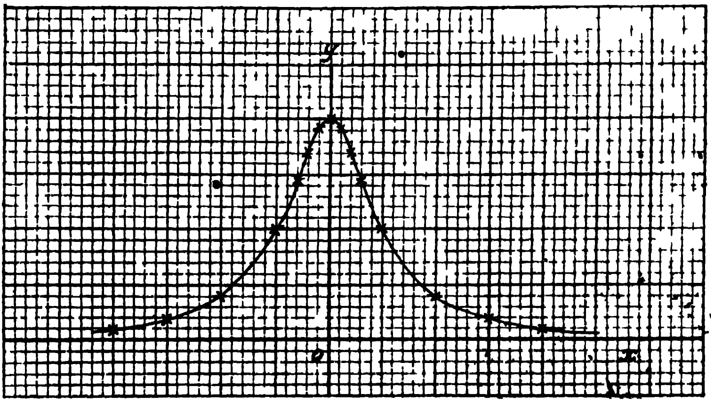
$$x^3 - 15x - 2 = 0.$$

The printed graph does not give the exact form of the curve given by $y = x^3 - 15x - 2$, for the x and y units are not equal. The curve shown is the real curve stretched uniformly in a direction parallel to Ox.

366. *Draw the graph of* $y = \dfrac{4}{x^2+1}$

When

$x=$	0	± 1	± 2	± 3	± 4	± 5	± 10
$y=$	$\frac{4}{1}$	$\frac{4}{2}$	$\frac{4}{5}$	$\frac{4}{10}$	$\frac{4}{17}$	$\frac{4}{26}$	$\frac{4}{101}$
$y=$	4	2	$\cdot 8$	$\cdot 4$	$\cdot 24$	$\cdot 15$	$\cdot 04$



In order to find the shape of the curve near Oy, we must find a few points near that line.

When

$x=$	$\cdot 2$	$\cdot 4$	$\cdot 6$
$y=$	$\frac{4}{1\cdot 04}$	$\frac{4}{1\cdot 16}$	$\frac{4}{1\cdot 36}$
$y=$	$3\cdot 85$	$3\cdot 45$	$2\cdot 94$

(This last line is conveniently written down by means of a slide rule.)

Plotting these points and joining them by an even curve, we have the graph as shown in the diagram.

367. *Draw the curve* $y=\dfrac{3}{2x-5}$.

When

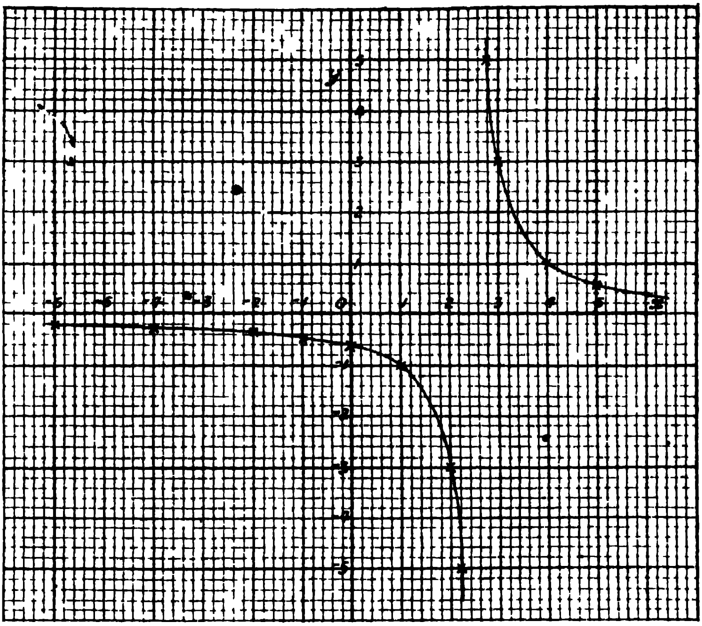
$x=$	-10	-6	-4	-2	-1	0
$y=$	$-\frac{3}{25}$	$-\frac{3}{17}$	$-\frac{3}{13}$	$-\frac{3}{9}$	$-\frac{3}{7}$	$-\frac{3}{5}$
$y=$	$-0\cdot 12$	$-0\cdot 18$	$-0\cdot 23$	$-0\cdot 33$	$-0\cdot 43$	$-0\cdot 6$
$x=$	1	2	3	4	5	10
$y=$	$-\frac{3}{3}$	$-\frac{3}{1}$	$\frac{3}{1}$	$\frac{3}{3}$	$\frac{3}{5}$	$\frac{3}{15}$
$y=$	-1	-3	3	1	$0\cdot 6$	$0\cdot 2$

To determine the shape of the curve between the points where $x=2$ and $x=3$, we must find some more points.

When

$x=$	$2\cdot 2$	$2\cdot 4$	$2\cdot 5$	$2\cdot 6$	$2\cdot 8$
$y=$	$-\frac{3}{0\cdot 6}$	$-\frac{3}{0\cdot 2}$	$\frac{3}{0}$	$\frac{3}{0\cdot 2}$	$\frac{3}{0\cdot 6}$
$y=$	-5	-15	∞	15	5

Plotting these points and joining them by an even curve, we see that there are two separate branches.



368. *Draw the graph of* $y^2 = x(x^2 - 16)$.

$$y^2 = (x - 4)x(x + 4).$$

If $x < -4$, y^2 is negative, and we have no real values of y.

Also if $x > 0$ and < 4, ...

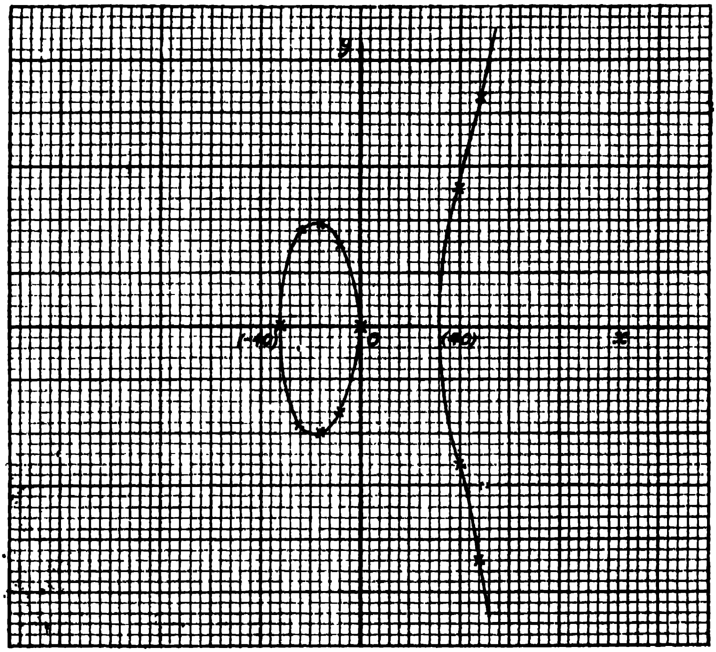
When					
x -	-4	-3	-2	-1	0
$x^2 - 16 =$	0	-7	-12	-15	-16
$x(x^2 - 16)$ -	0	21	24	15	0
$y =$	0	$\sqrt{21}$	$\sqrt{24}$	$\sqrt{15}$	0
$y =$	0	± 4.58	$+4.90$	$+3.87$	0

(The sq. roots can be obtained by means of a slide rule.)

Hence we see that this part of the curve is an oval in the 3rd and 4th quadrants.

When

$x^2-16=$	0		20	33	48
$x(x^2-16)=$	0	45	120	231	384
$y=$	0	$\sqrt{45}$	$\sqrt{120}$	$\sqrt{231}$	$\sqrt{384}$
$y=$	0	$\pm 6{\cdot}71$	± 11	$\pm 15{\cdot}2$	$\pm 19{\cdot}6$



Thus we see that the curve consists of two separate parts, as shown in the diagram.

369. *Draw a sketch of the curve given by the equation*

$$y=\frac{3}{(x-1)(x-4)}.$$

When

$x=$	-10	-5	-4	-3	-2	-1	0	1
$y=$	$\frac{3}{11\cdot14}$	$\frac{3}{6\cdot9}$	$\frac{3}{5\cdot8}$	$\frac{3}{4\cdot7}$	$\frac{3}{3\cdot6}$	$\frac{3}{2\cdot5}$	$\frac{3}{1\cdot4}$	$\frac{3}{0}$
$y=$	$\frac{\cdot2727}{14}$	$\frac{\cdot5}{9}$	$\frac{3}{40}$	$\frac{\cdot75}{7}$	$\frac{1}{6}$	$\cdot3$	$\cdot75$	∞
$y=$	$\cdot02$	$\cdot056$	$\cdot075$	$\cdot11$	$\cdot17$	$\cdot3$	$\cdot75$	∞

When

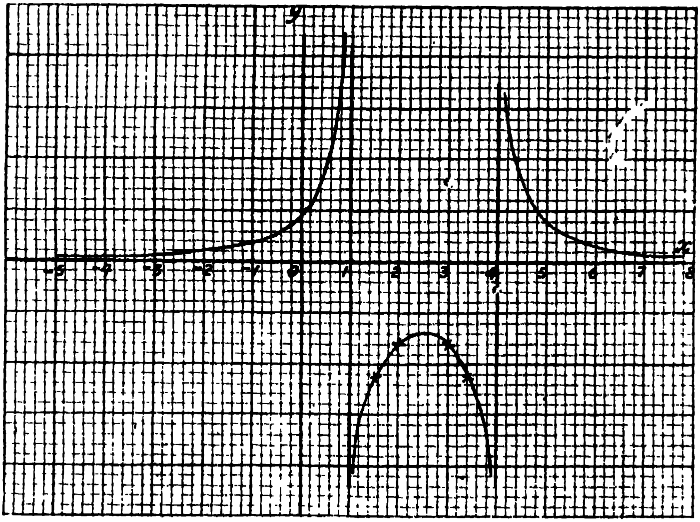
$x=$	2	3	4	5	6	7	8	10	100
$y=$	$-\frac{3}{1\cdot2}$	$-\frac{3}{2\cdot1}$	$\frac{3}{3\cdot0}$	$\frac{3}{4\cdot1}$	$\frac{3}{5\cdot2}$	$\frac{3}{6\cdot3}$	$\frac{3}{7\cdot4}$	$\frac{3}{9\cdot6}$	
$y=$	$-1\cdot5$	$-1\cdot5$	∞	$\cdot75$	$\cdot3$	$\cdot17$	$\frac{\cdot75}{7}$	$\frac{\cdot5}{9}$	
$y=$							$\cdot11$	$\cdot056$	

The values of x from 1 to 4 do not give us sufficient information as to the curve thereabouts, so we must tabulate for some more points.

When

$x=$	$1\cdot2$	$1\cdot5$	$2\cdot5$	$3\cdot5$	$3\cdot8$
$y=$	$-\frac{3}{\cdot2\times2\cdot8}$	$-\frac{3}{\cdot5\times2\cdot5}$	$-\frac{3}{1\cdot5\times1\cdot5}$	$-\frac{3}{2\cdot5\times\cdot5}$	$-\frac{3}{2\cdot8\times\cdot2}$
$y=$	$-\frac{\cdot75}{\cdot14}$	$-\frac{\cdot60}{\cdot25}$	$-\frac{\cdot20}{\cdot15}$	$-\frac{\cdot60}{\cdot25}$	$-\frac{\cdot75}{\cdot14}$
$y=$	$-5\cdot4$	$-2\cdot4$	$-1\cdot3$	$-2\cdot4$	$-5\cdot4$

Plotting these points, we see that the curve consists of three branches, as shown in the diagram.



370. Symmetry. If the equation of a curve contains only even powers of x, the curve is symmetrical about the axis of y; for changing x to $-x$ does not alter the value of y. Similarly, if the equation contains only even powers of y, the curve is symmetrical about the axis of x. (See Arts. 366, 368.)

If changing x to $-x$ alters the sign, but not the numerical value of y, there is symmetry in the 1st and 3rd quadrants, and also in the 2nd and 4th.

Examples. LIV.

In drawing the graphs of curves, the units employed should be stated on the sheet containing the graph; and the tabulations should be full and arranged so that the working can be easily checked.

Draw the graphs of the following equations:

1. $y=x^3+2$. 2. $y=x^3+2x$. 3. $y=\dfrac{1}{x+1}$.

4. $y^2=x^3$. 5. $y^2=x^3+4$. 6. $y=\dfrac{x-2}{x-5}$.

7. $y=\dfrac{3}{x^2-1}$. 8. $y=\dfrac{5}{3x-4}$. 9. $y=\dfrac{5}{x^2}$.

10. $y=\dfrac{5}{x^2+4}$. **11.** $y=(x-1)(x-2)(x-3)$. **12.** $y^2=x(x^2-9)$.

13. $y=\dfrac{5}{x(x-4)}$. **14.** $y=\dfrac{(x-3)(x-2)}{(x-4)(x-1)}$. **15.** $y=\dfrac{(x-3)(x-4)}{(x-2)(x-1)}$.

16. Plot the curve $y=x^3-6x+5$, and hence solve the equation

$$x^3-6x+5=0.$$

17. Plot the curve $y=x^3+2x^2-5x-6$, and hence solve the equation

$$x^3+2x^2-5x-6=0.$$

18. Plot the curve $y=x^3+6x-7$ for the following values of x,

$$-4,\ -3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4.$$

What do you deduce as to the roots of the equation $x^3+6x-7=0$?

19. Solve the equation $x^3-10x=7$.

20. Plot the curve $y=x^3-3x^2-10x+20$ for the following values of x, $-3,\ -2,\ -1,\ 0,\ 1,\ 2,\ 3,\ 4$, and hence obtain two roots of the equation. Deduce the value of the third root.

21. Plot the curves $y=x^3$ and $y=x^2-5$, and hence find a root of the equation $x^3-x^2+5=0$. What do you deduce as to the other roots?

22. Using as large units as convenient, plot in the same diagram between the points (0, 0), (1, 1) the graphs of $y=x^n$ when $n=3,\ 2,\ 1,\ \frac{1}{2},\ \frac{1}{3}$.

CHAPTER LV.

MISCELLANEOUS EXAMPLES.

LV. a.

1. By successive divisions by 5 obtain $\frac{1}{5}, \frac{1}{5^3}, \frac{1}{5^5}, \frac{1}{5^7}$ in decimals, and hence find the value of π correct to 4 decimal places from the formula

$$\frac{\pi}{4}=4\left(\frac{1}{5}-\frac{1}{3}\cdot\frac{1}{5^3}+\frac{1}{5}\cdot\frac{1}{5^5}-\frac{1}{7}\cdot\frac{1}{5^7}\right)-\frac{1}{239}.$$

2. If £·001 is taken to be a farthing, what percentage of a farthing is the error? If y shillings $+z$ pence $=$ £0·775, write down the values of y and z.

3. Solve the equation $\sqrt{x^2+72}=x+6$.

4. A certain old book is now worth £50, and is increasing in value at the rate of £5 a year. Assuming the value to continue rising at the same rate, find an expression for its value at any future time, and draw a diagram to show the value at any time. From your figure show the value of the book in 25 years.

5. A man bought cloth for £12. If he had bought 1 yard less for the same money, each yard would have cost him one shilling more. How many yards did he buy?

6. A number of two digits and the same number reversed are added together, and the sum is 121. If the digits differ by 3, what is the number?

7. A manufacturer has priced certain lathes. The largest sells at £175, and the smallest at £40. He wishes to increase his prices so that the largest will sell at £200 and the smallest at £50. Assuming that the new price P and the old price Q are connected by the relation $Q=a+bP$, find the values of a and b, and, to the nearest pound, the new prices of lathes valued at £150, at £125. 10s. and at £78.

8. A rectangle has sides of length a and b. The error in measuring these is x per cent. and is very small. Prove that the error in the area is 0 or $\frac{abx}{50}$.

9. If 40 men can be arranged in a hollow square 2 deep, find the number in the front.

10. A man sets out to walk from A to B, a distance of 16 miles, at the rate of $3\frac{1}{2}$ miles an hour. Three quarters of an hour afterwards a man sets out from B to meet him at the rate of 4 miles an hour. When and where do they meet?

LV. b.

1. The sum of the first n even numbers $=\left(1+\frac{1}{n}\right)$ times the sum of the first n odd numbers.

2. Solve the equations $x^2y+xy^2-2xy=6$.

3. In the equation $3x^2+8xy+5y^2-12x-8y+1=0$, $x+h$, $y+k$ are substituted for x, y, and it is found that the coeffts. of x and y vanish in the new equation. Find the values of h and k.

4. If $x+y+z=0$, then $\Sigma(x^2)=-2\Sigma(xy)$ and $\Sigma\left(\frac{1}{x^2}\right)=\left\{\Sigma\left(\frac{1}{x}\right)\right\}^2$

5. The side of a square is 55 feet long. Find the dimensions of a rectangle whose perimeter is 2 feet longer than that of the square, and whose area is 1 sq. foot less.

6. Find the square of the series

$$1+\tfrac{1}{2}x+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot x^2+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\tfrac{5}{6}x^3+\dots \text{ to infinity, if } x<1.$$

7. Through the intersection of the straight lines whose equations are $2x-3y-7=0$ and $x-4y-1=0$ a straight line is drawn parallel to the axis of y. Find graphically its equation.

8. Express $\cdot03125$ as a power of 4.

9. An electric tram line with a single line of rails is $3\frac{1}{4}$ miles long. There are 13 trams available, and it is desired to arrange a 3-minute service of trams. How many crossing places must be provided, and what must be the speed of the trams?

10. Draw the graph of $y=x^2-3x+4$. Find the two points where it is cut by the graph of $y=mx$.

For what values of m are these coincident?

LV. c.

1. Plot the points $(1, 2)$, $(-3, 2)$, $(2, -1)$, $(-2, -3)$ on squared paper, taking the inch for unit. State the coordinates of the mid-points P, Q of the lines joining the first two points and the last two, and find the abscissa of the point where PQ cuts the axis of x.

2. If $v^2/r=g/289$, calculate v, having given that $r=4000$, $g=32\cdot2/5280$. Also show that $\dfrac{2\pi r}{v\times60\times60}=24$ approximately, where $\pi=3\cdot1416$.

3. Prove in any way that approximate values of $2^{\frac{1}{3}}$ and $2^{\frac{7}{12}}$ are $\frac{5}{4}$ and $\frac{3}{2}$ respectively. Find in each case whether the approximate or true value is the greater, and in which case the approximation is the closer.

4. Calculate $\log\dfrac{347\times\cdot231}{7\cdot62}$.

5. Solve the equation $(x+1)(x+2)(x-3)(x-4)=336$.

6. Give the remainder when $x^3-3x^2-18x+40$ is divided by $x-2$. Also when x^2+px+q is divided by $x-a$.

7. A man has 86 coins consisting of half-sovereigns, crowns, and florins. The sums of money in these coins are respectively proportional to 15, 13, 9. Find the numbers of the different coins.

8. Find a point C in the line AB (a inches long), so that the sum of the squares on AC and CB shall be equal to a given square (side b inches). Find AC to the nearest hundredth of an inch when $a=4$, $b=3$.

9. If $a=x^k$, $b=x^l$, $c=x^m$, express as a power of x the fraction a^pb^q/c^r.

10. Having given that $10^x=2$, calculate 10^{2x}, 10^{4x}, 10^{8x}, 10^{16x}; and hence show that x lies between $\frac{4}{16}$ and $\frac{5}{16}$.

Show without using tables that $\log_{10}2$ lies between $\cdot25$ and $\cdot32$.

LV. d.

1. Find the least fraction which is greater than $\frac{12}{17}$ and has its denominator greater by 13 than its numerator.

2. Find the sum of all numbers between 400 and 1000 which are divisible by 17.

3. Show that if all the terms of a G.P. are positive, the sum to infinity cannot be less than 4 times the 2nd term.

4. Find which term in the expansion of $(3x+5)^{20}$ has the greatest coefficient.

5. Find the value of $9^{\frac{1}{6}} \times 3^{\frac{1}{12}} \times 2^{\frac{1}{24}}$.

6. The number of births in a town is 25 in every thousand of the population annually, and the deaths 20 in a thousand. In how many years will the population double itself?

7. If $f(n)=\frac{1}{6}n(n+1)(2n+1)$, find $f(n+1)-f(n)$.

8. Find a positive value of x satisfying the equation $2^x=7\times 2^{\frac{1}{x}}$ approximately, so that the error does not exceed one per cent.

9. Solve the equation $2x^2+y^2-2xy-2y+1=0$ as an equation for y, and ascertain for what values of x, y is real.

10. If
$$(1+x)^n=1+n_1x+n_2x^2+n_3x^3+\dots,$$
find the value, when $n=9$, of $1-n_2+n_4-n_6+n_8$.

LV. e.

1. Evaluate $2^{-2}3^{\frac{1}{2}}x^{\frac{5}{6}}+2^{-1}3^{-\frac{1}{2}}x^{\frac{2}{3}}-10(27x)^{-\frac{1}{6}}$ when $x=64$.

2. Solve the equation $\sqrt{(x+\sqrt{x})}+\sqrt{(x-\sqrt{x})}=3\sqrt{x/(x+\sqrt{x})}$.

3. Prove that the cube of a number of n digits cannot have less than $3n-2$ nor more than $3n$ digits.

4. A walks m miles in n hours; B walks $7n$ miles in $\frac{1}{2}m$ hours; the difference of their rates of walking is $\frac{1}{2}$ mile per hour. Find the rate at which each walks.

5. Show that when $4a+4b+1=0$, the equations
$$x^2-2ax+b=0,\quad x^2-2bx+a=0,$$
have a common root.

6. If x, y, z are in H.P., prove that $\frac{x}{y+z}$, $\frac{y}{x+z}$, $\frac{z}{x+y}$ are in H.P.

7. A number is expressed by 2 digits, whose sum is 8, with a decimal point between them, and its double is one less than the number obtained by interchanging the digits: what is the number?

8. In the expansion of $(a+b)^{10}$ if $a+b=1$, show by logarithms that the first term > the sum of all the remainder if a is greater than (about) ·933.

9. The interest on a sum of P pounds for a certain time is I pounds and the discount at the same rate of (simple) interest is D pounds. Show that $\frac{1}{P}=\frac{1}{D}-\frac{1}{I}$.

10. A country trebles its population in a century. What is the increase per thousand in one year?

LV. f.

1. A and B are two stations on a line 90 miles apart. At the same instant one train passes through A towards B and another through B towards A, with different but constant speeds. They pass each other at C, and AC is 10 miles longer than BC; also, the first reaches B half an-hour before the second reaches A. Find their speeds by graphical methods.

2. Explain how to test multiplication of numbers by 'casting out the nines.' Two numbers A, B give remainders 5 and 7 respectively when divided by 9: find the remainder when their product is divided by 9.

3. Find the number of digits in the 250th term of $2+4+8+16+\ldots$.

4. Find log 83 to the base 19 as accurately as your tables permit.

5. The weight of a spherical shell is $\frac{7}{8}$ of what it would be if solid. Compare the inner and outer radii; and if the inner be increased by one half, find in what ratio the weight is reduced. [Weight of sphere $\propto$ radius^3.]

6. A man walks a third as far again as a boy in a given time. His stride is 5 centimetres longer than the boy's, and he takes 125 strides per minute, while the boy takes only 100. At what rate in kilometres per hour does each walk?

7. In how many ways could a party of 5 scouts be selected from 20 men? In how many ways could the 20 men be formed into 4 parties of 5 scouts, to proceed in different directions?

8. Sum to n terms the series whose n^{th} term is $an+2^{n-1}$.

9. What values must a, b, c have that $3x^2+10x+3$ may be equal to $a(x-b)(x-c)$ for all values of x?

10. If $a+b+c=p$, then $a(b+c)+b(c+a)+c(a+b)$ cannot exceed p^2.

LV. g.

1. Find the value of $\dfrac{x+p}{q}-\dfrac{x+q}{p}$, when $x=\dfrac{q^2}{p-q}$.

2. Calculate to 4 significant figures:

$$\text{(i)}\ \frac{52{\cdot}45\times378{\cdot}4\times{\cdot}02086}{87{\cdot}32\times{\cdot}5844}.\qquad\text{(ii)}\ (1{\cdot}246)^{1{\cdot}125}.$$

3. Simplify $\sqrt{3+\sqrt{5}}+\sqrt{3-\sqrt{5}}$, and find its value to 3 places.

4. A and B start together in a long distance race. For a quarter of an hour A goes at the rate of x yds. per second, and B $2x$ miles per hour; and then A is leading by 100 yds. Find x.

5. If $\dfrac{a}{b}=\dfrac{c}{d}$, and x be a 3rd proportional to a and b, y to b and c, prove that the geometric mean of x and y = that of c and d.

6. A meteor is seen at a height of 450 feet above the ground, one second later it is at a height of 328 feet. and 2 seconds after the 1st observation it is at a height of 174 feet. Determine the values of a, b, c, if the formula at^2+bt+c represents for all values of t the height of the meteor t seconds after the 1st observation.

7. $f(x)=\text{A}\dfrac{(x-b)(x-c)}{(a-b)(a-c)}+\text{B}\dfrac{(x-c)(x-a)}{(b-c)(b-a)}+\text{C}\dfrac{(x-a)(x-b)}{(c-a)(c-b)}$.
Find $f(a)$, $f(b)$, $f(c)$.

8. Find by the Binomial Theorem the first 20 figures of $\frac{1}{99^5}$.

9. Expand $\log(2+x-x^2)$ in ascending powers of x, if $x<1$.

10. Prove by Mathematical Induction that $(3n+1)7^n-1$ is always divisible by 9.

LV. h.

1. Prove that $\frac{a+p}{b+p}$ lies between $\frac{a}{b}$ and $\frac{a+2p}{b+2p}$, where a, b, p, q are all positive quantities.

2. What is the necessary condition that the sum to infinity of the series $\frac{a^2}{b}+\frac{a}{c}+\frac{b}{c^2}\ldots$ may be finite? Find the sum when this condition is satisfied.

3. A certain football club consists of 20 members, of whom 11 can only play forward, and 9 can only play back. In how many ways can a team of 8 forwards and 7 backs be selected?

4. Find the coeff. of x^{17} in $\left(\frac{x^2}{2}-\frac{2}{x}\right)^{16}$.

5. If α, β are the roots of the equation $ax^2+bx+c=0$, construct the quadratic whose roots are $\frac{1-\alpha}{1+\alpha}$, $\frac{1-\beta}{1+\beta}$.

6. Find approximately by logarithms the value of the incommensurable root of $3^{2x}-5\times 3^x+6=0$.

7. Write down the first negative term in the expansion of $(1+x)^{\frac{27}{5}}$, x being positive.

8. If $x<a$, and $x<b$, resolve $\frac{1}{ab-(a+b)x+x^2}$ into partial fractions and expand each of them. Prove that the coeff. of x^{n-1} is $\frac{a^{-n}-b^{-n}}{b-a}$.

9. Prove that 14641 is a square number in any scale.

10. The consumption of coal by a locomotive $\propto$ the square of the velocity. When the speed is 16 miles an hour, the hourly consumption is 2 tons. If the price of coal be 10*s.* a ton, and the other expenses be 11*s.* 3*d.* an hour, find the most economical speed, and the least cost for a journey of 100 miles.

LV. k.

1. Find correct to 2 decimal places $\sqrt[3]{2\cdot718}$ and $\sqrt[5]{673\cdot8}$.

2. If x^3+2ax^2+bx+c is divisible by x^2+ax+d, find b and c in terms of a and d.

3. Find the sum of all the numbers between 600 and 2000 which are divisible by 7.

4. How many numbers are there consisting of five different digits arranged in ascending order of magnitude?

5. If $a+b+c=0$, then $2(a^5+b^5+c^5)=5abc(a^2+b^2+c^2)$.

6. Draw the graphs of $y^2=12x$ and $y=x+3$, and find where they meet.

7. Find two successive terms in the expansion of $(1+x)^{29}$ such that their coefficients are in the ratio $5:1$.

8. Given the height, the volume of a cylinder varies as the square of the radius. Find the volume of a cylinder whose radius is 5 inches, if the volume of a cylinder of the same height and radius 3 inches is 282·744 cubic inches.

9. Show that the sum of the series

$$\frac{5}{6}+\frac{5.7}{6.9}+\frac{5.7.9}{6.9.12}+\ldots \text{ to infinity}$$

is $3\sqrt{3}-2$. (A binomial expansion with index $-\frac{3}{2}$.)

10. Sum the series

$$\frac{1}{5}-\frac{1}{2}\cdot\frac{1}{5^2}+\frac{1}{3}\cdot\frac{1}{5^3}-\frac{1}{4}\cdot\frac{1}{5^4}+\ldots \text{ to infinity.}$$

LV. l.

1. Prove that if a number of 2 digits $=4$ times the sum of its digits, the reversed number $=7$ times the sum of the digits.

2. If $x+y:x-y=a:b$, express the ratio $x^2+y^2:x^2-y^2$ in terms of a and b.

3. Find the coefficient of x^{10} in $\{(1+x)(1+x^2)\}^8$.

4. Find the greatest value of $\dfrac{x+2}{4x^2+16x+25}$.

5. If $ma^2+nc^2:pa^2+qc^2=mb^2+nd^2:pb^2+qd^2$, then either $a:b=c:d$, or $m:n=p:q$.

6. Find 4 numbers in A.P. whose sum is 32, the harmonic mean between the 2nd and 3rd being $7\frac{1}{2}$.

7. The 1st and 2nd of the 3 digits by which a perfect square is expressed are 1 and $2n$ respectively; find the 3rd digit, and show that, in any scale, a perfect square is obtained by reversing the digits.

8. Solve $\dfrac{x}{a}-\dfrac{y}{b}=2$, $\sqrt{2a(b-y)}=\sqrt{b}(\sqrt{a}+\sqrt{x})$.

9. Show that

$$(1-x^2)^n=(1+x)^{2n}-2nx(1+x)^{2n-1}+\frac{2n(2n-2)}{1.2}x^2(1+x)^{2n-2}-\ldots.$$

10. Find the coefficient of x^{12} in the product of

$$\frac{1+x^3}{(1-x^2)(1-x)} \text{ and } 1-x+x^2.$$

LV. m.

1. A tricycle, going 5 miles an hour, passes a milestone, and 14 minutes afterwards a bicycle, going in the same direction at 12 miles an hour, passes the same milestone; find graphically when and where the bicycle will overtake the tricycle.

2. Solve the equation $x(x^2-4)=a(a^2-4)$.

3. If the sum of the first p terms of an A.P. $=0$, the sum of the next q terms $=\dfrac{-a(p+q)q}{p-1}$.

4. If a, b, c are in A.P.; a, mb, c in G.P.; prove that a, m^2b, c are in H.P.

5. Determine the scale of notation with highest radix which requires 4 digits to express the number five hundred; and find what the 4 digits will be.

6. Find the coefficient of x^8 in the expansion of $\frac{1}{1+x+x^2+x^3}$.

7. If $c = a - b$, and if c is very small compared with a and b, then $a^2b^2(a^2 - a^2x^2 + b^2x^2)^{-\frac{3}{2}} = a - 2c + 3cx^2$ nearly.

8. Sum the series $1 . 3 + 2 . 4 + 3 . 5 + \ldots + n(n+2)$.

9. Prove $\log_e a - \log_e x = \frac{a-x}{a} + \frac{1}{2}\left(\frac{a-x}{a}\right)^2 + \frac{1}{3}\left(\frac{a-x}{a}\right)^3 + \ldots$, if $\frac{x}{a}$ is a positive proper fraction.

10. Resolve into partial fractions $\frac{x^2 - x + 1}{(x^2+1)(x-1)^2}$.

LV. n.

1. Insert 19 Arithmetic and also 19 Harmonic means between 2 and 3, and show that if A be any arithmetic mean and H the corresponding harmonic mean, $A + \frac{6}{H} = 5$.

2. The number of permutations of *abbccc* in which the b's come together $= \frac{1}{3}$ of the whole number of permutations.

3. If a, b, c, d are in G.P., $\log_a N$, $\log_b N$, $\log_c N$, $\log_d N$ are in H.P.

4. Resolve into partial fractions $\frac{2b^2}{(x-a+b)(x-a-b)(x-a)}$.

5. Find the relation between a, b, and c in order that $x^3 + 3ax^2 + bx + c$ may be a cube.

6. Expand $(1-x)^{-2}$, $(1-x)^{-3}$, $(1-x)^{-4}$ giving the coefficient of x^n in each case.

7. Find sq. rt. of 23t314 in the scale of 12.

8. Draw a circle 2·6 inches in radius and take a point O 4·4 inches from its centre. Consider a straight line OPQ passing through O and cutting the circle in P, Q. Draw this line in 4 positions including the one in which P and Q coincide, and by measuring OP and OQ in each position, prove that the rect. OP, OQ is constant. Find an equation connecting r the radius of the earth, h the height of a mountain, and d the distance of the sea-horizon as seen from the mountain-top.

9. An approximation to $\frac{1}{365}$ is $\cdot 002\left(1 + \frac{1}{3} + \frac{1}{30} + \frac{1}{300}\right)$. To how many places is it correct?

10. The height of a mountain in feet is given by

$$H = 49000\left(\frac{R-r}{R+r}\right)\left(1 + \frac{T+t}{900}\right),$$

where R, r are the observed heights of the barometer in inches at the bottom and top of the mountain, and T, t are the observed temperatures

at the bottom and top. The following observations were made at the bottom and top of a certain mountain:

	Barometer.	Thermometer.
At bottom	29·60 in.	67°
At top	25·35 in.	30°

Find the height of the mountain and roughly check your result by any method which occurs to you.

LV. p.

1. If x is real, show that $\dfrac{x^2 - 4x - 20}{x - 7}$ cannot have a real value between 8 and 12.

2. The ratio of B's age to A's age is now 7 : 4, but in 10 years' time it will be 19 : 13. Find their ages.

3. Draw the graph of the equation $2x^2 + 5xy - 3y^2 - 5x + 6y - 3 = 0$, after resolving into factors the expression in x, y.

4. During any year the excess of births over deaths causes an increase of h per cent. of the population, and at the end of the year a fixed number A of people emigrate. Prove that at the end of n years a population P becomes $b^n\mathrm{P} - \dfrac{b^n - 1}{b - 1}\mathrm{A}$, where $b = 1 + \dfrac{h}{100}$.

5. Find, without assuming any formula, the number of different ways in which n men may stand in a row.

If two specified men are neither of them to be at either end of the row, show that the number of arrangements is $(n-2)(n-3)|\underline{n-2}$.

6. Eliminate a from the equations $x = \log_a b$, $y = \log_a c$.

7. Solve the equation $24^3 = 16^{x-1}$.

8. Prove by the Binomial Theorem that money at 5 per cent. compound interest more than doubles itself in 15 years.

9. The middle term of $(1+x)^{2n}$ is $\dfrac{1 . 3 . 5 \ldots (2n-1)}{|\underline{n}}(2x)^n$.

10. Prove $\dfrac{1}{1 . 2 . 3} + \dfrac{1}{3 . 4 . 5} + \dfrac{1}{5 . 6 . 7} + \ldots = \log_e 2 - \frac{1}{2}$.

LV. q.

1. Prove by Mathematical Induction that the sum

$$1 + 3 + 9 + 27 + \ldots \text{ to } n \text{ terms is } \tfrac{1}{2}(3^n - 1).$$

2. Two casks A and B contain mixtures of wine and water, A in the ratio of 7 to 4, and B in the ratio of 9 to 2. In what ratio must liquid be drawn from each cask to give a mixture in the ratio of 3 to 1?

3. A quantity of water contained in a cubical cistern is found to lose by evaporation ·04 of its volume in a day. The depth is 6 feet, and a cub. ft. of water weighs 1000 oz. Assuming the loss to be only through evaporation, what weight of water will be left at the end of 10 days?

4. If the expression $x(1 - 5x + 6x^2)^{-1}$ be expanded in powers of x, prove that the coeff. of x^r is $3^r - 2^r$. [Resolve into partial fractions.]

5. Expand $(1+x^2)^{-\frac{1}{2}}$ when $x>1$.

6. If $x+\dfrac{1}{x}=y$, express $x^3+\dfrac{1}{x^3}$ and $x^5+\dfrac{1}{x^5}$ in terms of y.

7. The duration of a railway journey varies directly as the distance and inversely as the velocity. The velocity varies directly as the sq. root of the quantity of coal used per mile and inversely as the number of carriages. In a journey of 25 miles in half an hour with 18 carriages, 10 cwt. of coal is required. What will be consumed in a journey of N miles in 20 minutes with 20 carriages?

8. Write down the n^{th} term of the series 8, 15, 24, 35, ... and find the sum of n terms. [$8=3+5$, $15=3+5+7$.]

9. Expand $\log(x^2+4x+3)$ in ascending powers of x, supposing x to be less than 1.

10. If

$$(1-x)^{-1}(1-x^2)^{-p}(1-x^3)^{-q}(1-x^4)^{-r}=1+x+2(px^2+qx^3+rx^4)+\dots,$$

show that $$p=\tfrac{1}{2}q=\tfrac{1}{5}r=1.$$

LV. r.

1. How many square yards are there in a courtyard x ft. y inches long and y ft. x inches wide?

If a part of it x ft. x inches by y ft. y inches is covered with grass, how many tiles $\dfrac{x-y}{2}$ inches square will cover the remainder?

2. Show that one root is the square of the other in the equation

$$x^2-x=2+\sqrt{5}.$$

3. If $x=\log_a bc$, $y=\log_b ca$, $z=\log_c ab$, prove that $xyz=x+y+z+2$.

4. Find the value, when $x=\frac{1}{4}\sqrt{3}$, of

$$\frac{1+2x}{1+\sqrt{1+2x}}+\frac{1-2x}{1-\sqrt{1-2x}}.$$

5. B and C together can finish a piece of work in $\frac{1}{3}$ of the time that A would take; C and A together in $\frac{1}{6}$ of the time that B would take. Compare the times in which A and B can separately do the work.

6. If a, b, c, d are in continued proportion, $a^2+d^2>b^2+c^2$.

7. The hypotenuse of a right-angled triangle is less than the sum of the other two sides by 8 ft., and the area $=180$ sq. ft. Find all the sides.

8. There are 8 stations on a railway. How many different sorts of single tickets of each class must be printed so as to include all possible journeys on that line?

9. Draw the graph of $x^2-6x+41$ and find its minimum value.

10. Resolve into partial fractions $\dfrac{x+1}{(x-1)^2(x^2+4)}$.

LV. s.

1. Simplify $(27a^{\frac{3}{4}})^{\frac{1}{3}}\times(9a^{-\frac{3}{2}})^{-\frac{1}{2}}$.

2. If the sum of n terms of a series is an expression of the 2nd degree in n with no absolute term, the series is an A.P.

3. Solve the equation $41(x^{\frac{1}{2}}-1)^2=x+1$.

4. If $(a^2+b^2+c^2)(x^2+y^2+z^2)=(ax+by+cz)^2$, prove that $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$.

5. Given that the time in seconds of a beat of a pendulum is $\pi\sqrt{\frac{l}{g}}$, where l is the length in feet, find the length of a pendulum which beats seconds if $g=32\cdot2$ and $\pi=3\cdot142$.

6. If $x+2\left(x-\frac{1}{n}\right)+3\left(x-\frac{2}{n}\right)+4\left(x-\frac{3}{n}\right)\dots$ to $n+1$ terms $=0$, prove that $x=\frac{2}{3}$.

7. Simplify $(3\sqrt{3}-5)^{\frac{1}{2}}\div(2-\sqrt{3})$, and find its value to 3 places.

8. In how many ways can 10 boys be arranged so that a particular pair never sit next to each other (1) when they form a row, (2) when they form a ring?

9. A starts $37\frac{1}{2}$ minutes before B to walk $11\frac{3}{4}$ miles, the first $5\frac{1}{4}$ of which are uphill and the rest downhill. Downhill their speeds are respectively $\frac{1}{4}$ and $\frac{1}{2}$ mile per hour faster than uphill, and they finish their journey at the same moment, though A reaches the hilltop $22\frac{1}{2}$ minutes before B. What are their initial speeds?

10. If a, x, y, b are in A.P., and c^3, x, y, d^3 in G.P., show that
$$a+b=cd(c+d).$$

LV. t.

1. Draw on squared paper the line $2x+0\cdot7y=2\cdot5$ (unit = 1 inch). Find to 1 decimal place the length of the perpendicular to it from the origin.

2. Find r, if ${}^nP_r=120\,{}^nC_{n-r}$.

3. In a certain year a rise of 5·37 per cent. in the female population, and of 5·61 in the male, produced a rise of 5·5 per cent. on the whole. Compare the number of males and females at the beginning of the year, and also at the end.

4. Write down the continued product of the n factors
$$1+x_1,\ 1+x_2,\ 1+x_3,\ \dots 1+x_n,$$
and deduce the expansion of $(1+x)^n$.

5. Find the 12th term and middle term of $\left(a-\frac{1}{a}\right)^{16}$.

6. A pendulum 3 ft. long is observed to make 700 beats in 671 seconds. If the time of a beat is $\pi\sqrt{\frac{l}{g}}$ seconds, where l = length of pendulum in feet, find the value of g, supposing that $\pi=3\frac{1}{7}$.

7. Find $\sqrt[3]{128}$ to 5 places of decimals.

8. In how many years will £20 become £36 at 5 per cent. compound interest?

9. The tension of a string with one end fixed and the other attached to a moving body varies directly as the square of the body's velocity and inversely as the length of the string. If the string will just bear the strain of a body moving with velocity 10 feet a second when the length of string is 3 feet, find the greatest velocity of the same body which the same string shortened to one foot will stand.

10. If $\phi(x)=x+\frac{1}{x}$, find $\phi(y+\sqrt{y^2-1})$.

LV. u.

1. Show that approximately $1000001^2 : 1000000^2 = 1000002 : 1000000$.

2. Prove that $(2+\sqrt{3})^{\frac{3}{2}} + (2-\sqrt{3})^{\frac{3}{2}} = 3\sqrt{6}$.

3. Solve the equation $3^x - 3^{x-1} = 486$.

4. The birth-rate in a town is 20 in a thousand, the death-rate is 14 in a thousand annually; in how many years will the population double itself?

5. In how many different ways can 8 persons be formed into 2 groups?

6. By resolving into partial fractions expand the fraction $\dfrac{1}{x^2-5x+6}$ in a series:

(i) of the form $p_0 + p_1 x + p_2 x^2 + \dots$,

(ii) of the form $q_0 + \dfrac{q_1}{x} + \dfrac{q_2}{x^2} + \dots$,

and show that $q_{n+1} = 6^n p_{n-1}$.

7. If $f(x)$, any rational integral function of x, is divided by $(x-a)(x-b)$, the remainder is $x\dfrac{f(a)-f(b)}{a-b} + \dfrac{af(b)-bf(a)}{a-b}$.

8. Sum to n terms the series whose r^{th} term is $2r - 2^r$.

9. Find the number of digits in $2^{19} \times 3^{17}$, and the first 4 digits of the product

10. Find the coefficient of x^{4r} in the expansion of $\log(1+x+x^2+x^3)$, where x is a proper fraction.

LV. v.

1. The incomes of A and B are as 6 to 5, their expenditure as 3 to 2. Each saves £270 a year: find their incomes.

2. Find a value of x which will make $x^4 + 2ax^3 + b^2x^2 + c^3x + d^4$ a square.

3. Find the scale of notation in which the common number 1156 is represented by 961.

4. Which term of the series 1, 3, 5 ... is a mean proportional between the 23rd and 63rd? Find also how many terms of the series have their sum equal to one-fourth of the sum of the first 20 terms.

5. A man arranges to pay off a debt of £3600 by 40 annual instalments, which form an A.P. When 30 of the instalments are paid he dies, leaving a third of the debt unpaid. Find the value of the first instalment.

6. Expand $(\frac{2}{3}+x)^{\frac{7}{2}}$, and find which term has the greatest numerical coefficient.

7. Sum to infinity the series $x + \dfrac{x^3}{\lfloor 3} + \dfrac{x^5}{\lfloor 5} + \dots$.

8. Find the coefficient of x^6 in the expansion of $(1+x+x^2)^{-3}$.

9. A man's net income is 10 per cent. less than his gross income, which is derived partly from land and partly from personal property. He pays income tax at 8*d*. in the £ on the whole, and also local rates on the income from land. If his whole income had been derived from land, his net income would have been one ninth less. What are the local rates per £, and in what proportion is his gross income derived from the two kinds of property?

10. Draw the graph of $\dfrac{x^2-13x}{x^2+5x+6}$.

LV. w.

1. Simplify $\left(\dfrac{x^{p+q}}{x^q}\right)^p \div \left(\dfrac{x^q}{x^{q-p}}\right)^{p-q}$ and $2^{\frac{1}{2}} \cdot 16^{\frac{1}{4}} + 2^{-\frac{1}{2}} \cdot 4^{\frac{5}{4}} + 9^{\frac{7}{8}} \cdot 3^{\frac{1}{4}}$.

2. A rod whose length is 5 feet is broken into 3 pieces, out of which a right-angled triangle can be formed whose area is 120 sq. inches. Find the length of its sides.

3. If $xy+3 : xz+1 = y^2+3 : yz+1$, show that $y=x$ or $y=3z$.

4. Find the number of digits in 2^{64}, and if $a^1 . a^2 . a^3 \dots a^n = p$, find an equation giving n in terms of p and a.

5. Solve the equations $2^x = 8^{y+1}$, $9^y = 3^{x-9}$.

6. Decompose into partial fractions $\dfrac{6x^2-x-1}{(x^2+1)(x-2)}$.

7. A man spends on charitable objects an annual amount proportional to the square of his income, and spends £35 more when his income is £1200 than when it is £900 per annum. Find his charitable expenditure in each case.

8. Write down the 2 middle terms in the expansion of $\left(x-\dfrac{1}{x}\right)^{15}$, and find how many terms of $(1-3x)^{\frac{7}{3}}$ have positive signs.

9. Prove that the sum of two numbers is divisible by 11, if every digit which stands in an even place in one stands in an odd place in the other.

10. Sum the infinite series whose n^{th} term is $\dfrac{n}{\underline{|n-1}}$.

Prove that it is twice the square of the series

$$1+\frac{1}{2\,\underline{|1}}+\frac{1}{2^2\,\underline{|2}}+\frac{1}{2^3\,\underline{|3}}+\dots \text{ to infinity.}$$

LV. x.

1. Simplify $\dfrac{a-b}{a^{\frac{1}{3}}-b^{\frac{1}{3}}} \div (a^{\frac{1}{6}}-b^{\frac{1}{6}})^4$.

2. Express $\sqrt{158+60\sqrt{6}}$ as the sum of 2 surds, and find its value correct to 3 decimal places.

3. Sum the series $(5a+7b)+(8a+5b)+(11a+3b)+ \dots$ to 18 terms.

4. Show that if a and b are such that the sum of the squares of the 3 arithmetic means inserted between a and $b = a^2+b^2$, then the sum of the cubes of these means $= \frac{1}{2}(a+b)^3$.

5. $\log 2^{x+3} = 1{\cdot}2222$. Find x.

6. Find by logarithms the value of $\sqrt{13\sqrt{5} \div \sqrt{7}}$.

7. Find the number of ways in which 5 boys and 3 girls may be arranged in a row (1) so that the girls may be all together, (2) so that no two girls may be together.

8. If $(1-x)^{-m} = c_0 + c_1x + c_2x^2 + \dots$, prove that

$$c_0+c_1+c_2+\dots+c_k = \frac{\underline{|m+k}}{\underline{|m}\,\underline{|k}},$$

x being less than 1, and m, k being positive integers.

9. Find the least possible value of $4x+\dfrac{9}{x}$.

10. If the expression $x(1-5x+6x^2)^{-1}$ be expanded in powers of x, prove that the coefficient of x^r is 3^r-2^r.

LV. y.

1. If $x+a$ be a common measure of x^2+px+q and $x^2+p_1x+q_1$, prove that $a=(q-q_1)/(p-p_1)$.

2. A bicyclist started 3 minutes late to keep an appointment at a place 15 miles off. By going at a rate half a mile an hour faster than he need otherwise have done, he arrived at the proper time. At what rate did he ride?

3. In how many ways can 12 persons be divided into 3 sets of 4 each, one set to play lawn tennis, one to play croquet, and one to play bowls?

4. Find the coefficient of x^3 in $(32x-\frac{1}{2})^{20}$.

5. The number of shot in the base of a rectangular pile is 800, and the 6th course from the base contains 495. Find the number of shot in the complete pile.

6. $3^x=7175$. Find x.

7. Two casks, each containing 20 gallons, are filled, one with water, the other with spirit; x gallons are drawn from each cask, mixed, and the casks are again filled up with the mixture. When this is done a second time, it is found that the quantity of spirit in the 2nd cask is to the quantity of spirit in the other as 5 to 3. Find x.

8. If x be real, between what limits does $\dfrac{2x^2+6x+3}{2x+1}$ lie?

9. Find the coefficient of x^r in the expansion of $\dfrac{a-bx}{e^x}$.

10. How many years must elapse before a sum of money doubles itself at 3 per cent. compound interest?

LV. z.

1. If $(a^2+b^2)(x^2+y^2)=(ax+by)^2$, prove that $\dfrac{x}{a}=\dfrac{y}{b}$.

2. If a number consists of n figures, its square cannot contain more than $2n$ or less than $2n-1$ figures.

3. Find, from a table of common logarithms, the logarithm of 125 to the base $4\frac{1}{2}$.

4. If $\phi(n)$ denote n^3+3n^2+3n, find $\phi(n+1)$ and $\phi(-1)$.

5. Find the number of arrangements of the letters of the product of a^7b^3 so that all the b's come together.

6. Find the scale in which the common number 3963 becomes 30203.

7. Find the coefficient of x^5 in the product

$$(1-x)\left(1+nx+\frac{n\,.\,\overline{n+1}}{\lfloor 2}x^2+\frac{n\,.\,\overline{n+1}\,.\,\overline{n+2}}{\lfloor 3}x^3+\dots\ ad\ inf.\right).$$

8. If $y=x+x^2+x^3+x^4+\dots\ ad\ inf.$, prove that

$$x=y-y^2+y^3-y^4+\dots\ ad\ inf.$$

9. Sum to n terms the series whose n^{th} term is $8n^3 - 1$.

10. Prove $\log_e 3 = 1 + \frac{1}{3 \cdot 2^2} + \frac{1}{5 \cdot 2^4} + \frac{1}{7 \cdot 2^6} + \dots$ *ad inf.*

LV. aa.

1. If $\phi(x) = 1 - 3x + 2x^2$, prove that $\phi(x) \times \phi(-x) = 2\phi(x^2) - 1 + x^2$.

2. The numbers 114020 and 20 are in the scale of 9. Divide the first by the second and express the result in the common scale.

3. Show that $3(1 + 3 + 5 + \dots + 99) = 101 + 103 + 105 + \dots + 199$.

4. A and B together can do a piece of work in a certain time. If they each did half the work separately, A would have to work one day less, B 2 days more than before. Find the time in which A and B together do the work.

5. ABCDEFG is a regular octagon. Prove that the lengths of AB, AC, AD are proportional to 1, $\sqrt{2 + \sqrt{2}}$, $1 + \sqrt{2}$.

6. If $x = a + b + \frac{(a-b)^2}{4(a+b)}$, and $y = \frac{a+b}{4} + \frac{ab}{a+b}$, show that

$$(x-a)^2 - (y-b)^2 = b^2.$$

7. Find 2 numbers such that their sum, their difference, and the sum of their squares are proportional to 7, 1, 75.

8. The side BC of a $\triangle$ABC is produced to P, where BC $= a$ and CP $= x$. Compare the areas of the $\triangle^{\text{s}}$ ABC, ABP.

The sides BC, CA, AB of a $\triangle$ABC are produced to P, Q, R, where CP $=$ AQ $=$ BR $= x$. If BC $= a$, CA $= b$, AB $= c$, show that the ratio of the $\triangle$PQR to the $\triangle$ABC is

$$[(a+x)(b+x)(c+x) - x^3] : abc.$$

9. To do a piece of work a contractor can employ two classes of workmen, whose wages are in the ratio of 17 to 13. If he employs the higher paid men (who work the faster), he pays £148. 15*s*. in wages, being £10. 10*s*. less than the lower paid men would cost him. Compare their rates of work.

10. A dam has to be built of material g times as heavy as water; and its thickness in feet T at any depth d feet is calculated from the formula $T = g^{-\frac{1}{2}} d$. When the material is 2·3 times as heavy as water, what must the thickness be at a depth of 8 feet?

LV. bb.

1. Find the value of $\left(\frac{x+a}{x-a}\right)^2 - \frac{x}{3a}$ when $x = a(1 + 2\sqrt{3})$.

2. Simplify $\sqrt{(6 - \sqrt{17 - 12\sqrt{2}})}$.

3. Transforn 123456 from scale 10 to scale 7

4. Find $\sqrt[5]{242}$ correct to 5 places by the Binomial Theorem.

5. The general term of $(1 - 4x)^{-\frac{3}{2}}$ is $\frac{\lfloor 2r+1}{(\lfloor r)^2} x^r$.

6. Expand $\frac{5 - 10x}{2 - x - 3x^2}$, and give the general term.

7. Solve the equation $3^{2x} . 5^{3x-4}=7^{x-1} . 11^{2-x}$.

8. Find the values of x and y which satisfy the equations $y=x+2$, $y=bx+3$. Drawing graphs of the equations on squared paper, indicate the values of x and y which satisfy both equations (i) when $b=0{\cdot}7$; (ii) when $b=0{\cdot}8$; (iii) when $b=0{\cdot}9$. Is it possible to find values of x and y which satisfy the two equations when $b=1$?

9. The height h of the eye above the sea, and the distance d of the horizon, are connected by the equation $h^2+2h\mathrm{R}=d^2$, where R is the earth's radius. Taking $\mathrm{R}=3960$ miles, find, in feet, the height above the sea at which the distance of the horizon is 5 miles.

10. The population of Ireland was 4,458,775 at the census of 1901, and decreased so that in x years from the census it fell to $(0{\cdot}9477)^x$ of the population at the census. Estimate the population $2\frac{1}{4}$ years after the census to the nearest thousand.

LV. cc.

1. Show that $2(ab+cd)$ cannot be greater than $a^2+b^2+c^2+d^2$.

2. Find, by logarithms, the cube root of $3{\cdot}2$ and that of $\frac{2}{3}$.

3. Find the range of values of a for which the expression

$$(a-2)x^2+2(2a-3)x+5a-6$$

may have real factors.

4. Two men are walking with the same speed in opposite directions along a path parallel to a railway; a train passes one of them in a time t and the other in a time T; show that the train will describe a distance equal to its own length in a time which is the harmonic mean between T and t.

5. Find the product of $1+x+x^2+\dots+x^{2n-1}$

and $1-x+x^2-\dots-x^{2n-1}$.

6. Find within what limits the values of x must be when $x^2-7x+12$ is negative. Verify your result.

7. If $a^2+b^2=11ab$, show that

$$\log\frac{a-b}{3}=\frac{1}{2}(\log a+\log b).$$

8. An examiner has given marks to papers; the highest number of marks is 185, the lowest 42. He desires to change all his marks according to a linear law converting the highest number of marks into 250 and the lowest into 100; show how he may do this, and state the converted marks for papers already marked 60, 100, 150. Use squared paper, or algebra, as you please.

9. A is the horizontal sectional area of a vessel in square feet at the water level, h being the vertical draught in feet.

A	14,850	14,400	13,780	13,150
h	23·6	20·35	17·1	14·6

Plot on squared paper and read off and tabulate A for values of h, 23, 20, 16.

If the vessel changes in draught from 20·5 to 19·5, what is the diminution of its displacement in cubic feet?

10. Work the following three exercises as if in each case one were alone given, taking in each case the simplest supposition which your information permits:

(i) The total yearly expense in keeping a school of 100 boys is £2,100; what is the expense when the number of boys is 175?

(ii) The expense is £2,100 for 100 boys, £3,050 for 200 boys; what is it for 175 boys?

(iii) The expenses for three cases are known as follows:

£2,100 for 100 boys, £2,650 for 150 boys, £3,050 for 200 boys. Wh the probable expense for 175 boys?

If you use a squared paper method, show all three solutions together.

LV. dd.

1. If pv^k is constant; and if $p=1$ when $v=1$, find for what value of v, p is 0·2. Do this for the following values of k, 0·8, 0·9, 1·0, 1·1. Tabulate your answers.

2. Solve the equation $(x-2)(x-4)=n$, and write down four values of n, for each of which at least one of the values of x will be a positive whole number.

3. Find the range of values of a, for which both the roots of the equation $x^2-2ax+a^2-1=0$ are less than 4 and greater than -2.

4. A certain quantity y equals the sum of two quantities, one of which varies directly as x, and the other inversely as x; given that $y=2$ when $x=-1$, and that $y=4$ when $x=2$, find y in terms of x.

5. Show by the Binomial Theorem that $\left(\frac{9}{10}\right)^{\frac{4}{5}}=0·919166$, and verify the result by a logarithmic calculation. If you find a small difference between the two results, what explanation (of a general kind) can be given of the difference?

6. Compute $2·307^{0·65}$ and $23·07^{-1·25}$, using logarithms.

7. If $w=144\{p_1(1+\log r)-r(p_3+10)\}$, and if $p_1=100$, $p_3=17$; find w for the four values of r, $1\frac{1}{2}$, 2, 3, 4. Tabulate your answers.

8. Express $\dfrac{x-13}{x^2-2x-15}$ as the sum of two simpler fractions.

9. Suppose s the distance in feet passed through by a body in the time t seconds is $10t^2$. Find s when t is 2, when t is 2·01, and when $t=2·001$. What is the average speed in each of the two short intervals of time after $t=2$? When the interval of time is made shorter and shorter, what does the average speed approximate to?

10. $z=ax-by^3x^{\frac{1}{2}}$.

If $z=1·32$ when $x=1$, and $y=2$;
and $z=8·58$ when $x=4$, and $y=1$; find a and b.

Then find z when $x=$ [illegible] and $y=0$

LV. ee.

1. If n be a positive number, show that $\dfrac{1}{\sqrt{(n^2+1)}-n}$ is greater than $2n$.

2. From the logarithm tables find the value of the fifth root of 1·175.

3. If x is a positive number less than unity, find for what values of x the product $x(1-x)$ is less than $\frac{1}{6}$. With the same restriction on the values of x, if we write n instead of 6, show that n must be greater than 4.

4. Without using a table of logarithms, find the characteristic in the following cases:

(i) of the common logarithm of $\frac{1}{125}$.

(ii) .... $\sqrt[3]{\frac{3}{7}}$.

(iii) of the logarithm to the base 3 of 776.

(iv) .. . . 0·0776.

5. Compute, using logarithms,

$$\sqrt[3]{37\cdot24},\ \sqrt{3\cdot724},\ 372\cdot4^{2\cdot43},\ 0\cdot3724^{-2\cdot43}.$$

Find the third result to the nearest thousand.

6. Find within what limits the value of x must be when $\dfrac{x}{6}-\dfrac{1}{x}$ is positive. Verify your result.

7. A firm is satisfied from its past experience and study that its expenditure per week in pounds is

$$120+3\cdot2x+\frac{C}{x+5}+0\cdot01C,$$

where x is the number of horses employed by the firm and C the usual turnover.

If C is 2150 pounds, find for various values of x what is the weekly expenditure, and plot on squared paper to find the number of horses which will cause the expenditure to be a minimum.

8. If $A=P\left(1+\dfrac{r}{100}\right)^n$ and if $A=3P$ when $r=3\frac{1}{2}$, find n.

9. The New Zealand Pension law for a person who has already lived from the age of 40 to 65 in the colony is:

If the private income I is not more than £34 a year, the pension P is £18 a year. If the private income is anything from 34 to 52, the pension is such that the total income is just made up to 52. If the private income is 52 or more there is no pension.

Show on squared paper, for any income I, the value of P, and also the value of the total income. If a person's private income is say £50, how much of it has he an inducement to give away before he applies for a pension?

Show on the same paper the total income, if the pension were regulated according to the rule

$$P=18-\frac{9}{26}\cdot I.$$

10. Express $\dfrac{3x+2}{x^2-3x-4}$ as the sum of two simple fractions.

11. The following are the areas of cross section of a body at right angles to its straight axis:

A in square inches	250	292	310	273	215	180	135	120
x inches from one end -			41	70	84	102	130	145

Plot A and x on squared paper. What is the probable cross section at $x=50$? What is the average cross section and the whole volume?

12. The following table records the heights in inches of a girl A (born Jan. 1890) and a boy B (born May 1894). Plot these records. The intervals of time may be taken as exactly 4 months.

Year - -	1900	1901			1902			1903
Month - -	Sept.	Jan.	May	Sept.	Jan.	May	Sept.	Jan.
A - - -	54·8	55·6	56·6	58·0	59·2	60·2	60·9	61·[illegible]
B - - -	48·3	49·0	49·8	50·6	51·5	52·3	53·1	53·9

Find in inches per year the *average* rates of growth of A and B during the given period. At about what age was the growth of A most rapid? State this rate; divide it by her average rate.

ANSWERS TO THE EXAMPLES.

PART I.

I. a. (p. 2).

1. $7x$. **2.** $2a$. **3.** a. **4.** $4x$. **5.** $7x$. **6.** 0. **7.** $8ab$.
8. $5ab$. **9.** 0. **10.** $4xy$. **11.** $6xy$. **12.** $5ab$. **13.** $5abc$. **14.** $12x$.
15. $9ab$. **16.** $22ab$. **17.** $16a$. **18.** $14abc$. **19.** $5a$. **20.** $15x$.
21. 16. **22.** 32. **23.** 4. **24.** 3·2. **25.** 6. **26.** 20.
27. 2. **28.** 8. **29.** $1\frac{1}{2}$. **30** $\frac{1}{8}$. **31.** $\frac{1}{4}$. **32.** 1·25.
33. 3. **34.** 9. **35.** 5. **36.** $6\frac{1}{2}$. **37.** 7·2. **38.** 4·8.
39. 2. **40.** 4. **41.** 2·5. **42.** ·8. **43.** ·2. **44.** ·008.

I. b. (p. 3).

1. $x+2$. **2.** $x-3$. **3.** $3x$ pence, $7x$ pence, $11x$ pence, ax pence

4. $20x$, $2x$, $8x$, $10x$, $240x$. **5.** $2x$ miles, $7x$ miles, $\frac{x}{2}$ miles, ax miles.

6. $3x$, $36x$. **7.** $\frac{x}{12}$, $\frac{x}{36}$. **8.** $2x$, $24x$. **9.** $\frac{x}{7}$, $\frac{12x}{7}$. **10.** $16x$, xy.

11. $240x+12y$. **12.** xy pence, $\frac{xy}{12}$ shillings. **13.** $144x$.

14. $\frac{}{144}$. **15.** $10x$, $100x$, $1000x$, $\frac{x}{1000}$.

16. $\frac{x}{10}$, $\frac{x}{100}$, $\frac{}{1000}$, $\frac{}{1000,000}$. **17.** $2x$, $6x$, $14x$, $2ax$, x, $3x$, $\frac{7x}{2}$.

18. $(y-x)$£. **19.** $(x-y)$£. **20** $(x+y)$£.

I. c. (p. 6).

4. 9. **5.** 64. **6.** 32. **7.** x^3.
8. a^5. **9.** a^3x^2. **10.** a^2b^3c. **11.** $12ab$.
12. $20a^5$. **13.** $36a^2b^2c^2$. **14.** $84a^4y^3z$. **15.** x.
16. x^3. **17.** $4a$. **18.** x^6. **19** 25.
20. x^6. **21.** a^8b^2. **22.** $16x^6y^8$. **23** x^6.
24. a^3y^9. **25.** $8a^6y^{12}$. **26.** x^2. **27.** $2a$.
28. $3a^2$. **29.** 6. **30.** $3b$. **31.** x^2.
32. x. **33.** $3b^2c$. **34.** $\frac{1}{4}ab$. **35.** 13.
36. 25. **37.** 25. **38.** 49. **39.** 24.
40. 4. **41.** 1. **42.** 3. **43.** 144.
44. 64. **45.** 2. **46.** 4.

I. d. (p. 7).

1. 15. 2. 9. 3. 1. 4. 49. 5. 27.
6. 100. 7. 9. 8. 7. 9. 81. 10. 500.
11. 98. 12. 11. 13. 14. 14. 36. 15. 720.
16. 6. 17. 9. 18. 48. 19. 16. 20. 32.
21. 3. 22. 1. 23. 3. 24. 8. 25. 1.
26. 8. 27. $\frac{4}{15}$. 28. 6. 29. 16. 30. 168.
31. 16. 32. 24. 33. 0. 34. 0. 35. 0.
36. 2. 37. 0. 38. 0. 39. 1. 40. $\frac{3}{16}$.
41. 0. 42. 2. 43. 2. 44. $1\frac{1}{3}$.

II. a. (p. 9).

1. 2. 2. -2. 3. 4. 4. -5. 5. -18.
6. -4. 7. $2a$. 8. $-2a$. 9. $-6a$.
10. $2a$. 11. $6x$. 12. $6x$. 13. $4a^2$.
14. $-14x^2$. 15. $-3x^2$. 16. $-7a^2$. 17. $-3a^2$.
18. $4ab$. 19. $-12ab$. 20. $-2ab$. 21. $-7ab$.
22. $5xy$. 23. $-9a^2b$. 24. 0. 25. $-4ab$.
26. -9. 27. $3x$. 28. $-3ab$. 29. $-12abc$.
30. $-2abc$. 31. $-7xy$. 32. $4abc$. 33. $-10abc$.
34. $3x$. 35. $-3x$. 36. $3x$. 37. $-2x^3$.
38. $-5x$. 39. $-20x$. 40. $4x$. 41. $-9x^2$.

II. c. (p. 12).

1. 27. 2. -9. 3. -1. 4. 7. 5. 21.
6. -15. 7. 4. 8. -3. 9. 2. 10. 4.
11. 3. 12. 0. 13. -1. 14. 0. 15. -13.
16. 0. 17. $\frac{1}{2}$. 18. 0. 19. 4. 20. 2.
21. 0. 22. 18. 23. $\frac{1}{14}$. 24. $\frac{1}{6}$. 25. 122.
26. 0. 27. 0. 28. 0. 29. -56. 30. -89.
31. 106. 32. -11. 33. 7840. 34. 9.
35. $1\frac{4}{15}$. 36. 45. 37. 33. 38. 30.
39. 9, 4, 1, 0, 1, 4. 40. -10, -8, 10, 44, 94. 41. 4, $2\frac{1}{2}$, 3, $5\frac{1}{2}$, 10.

II. d. (p. 14).

1. 7. 2. -6. 3. 0. 4. $-13a$. 5. $5bc$.
6. $-10x^2y+xy^2$. 7. $3x^2-8xy-3y^2$. 8. $8a$. 9. $2a$. 10. $2a^2$.

III. a. (p. 16).

1. 8. 2. 2. 3. 8. 4. 10. 5. -1. 6. 5. 7. 0.
8. 16. 9. 16. 10. -9. 11. 0. 12. 0. 13. 19. 14. 4.
15. $8a$. 16. $4a$. 17. 0. 18. $12a$. 19. $-a$. 20. a.
21. $-3a$. 22. $3a$. 23. $5a^2$. 24. 10. 25. $-3x^2$. 26. 0.

III. b. (p. 18).

1. -3. 2. z. 3. -6. 4. -1. 5. 0. 6. 0.
7. x. 8. $-6x$. 9. $2x$. 10. $-4x$. 11. $7a$. 12. $-a$.
13. $-9a$. 14. $4a$. 15. $5a$. 16. $-2x^2$.
17. $2abc$. 18. 0. 19. $\frac{3x}{2}$. 20. $\frac{x}{2}$.
21. $-\frac{5x}{2}$. 22. $\frac{5x}{2}$. 23. $2a^2+2a$. 24. $3a^2-3a$.
25. $-6x^2-2x$. 26. $-2x^2+x$. 27. $\frac{3x}{4}$. 28. $\frac{x}{4}$.
29. $-\frac{x}{4}$. 30. $-\frac{3x}{4}$. 31. $\frac{5x}{8}$. 32. $\frac{x}{8}$.
33. $\frac{1}{4}xyz$. 34. $-\frac{x}{6}$. 35. $-\frac{x^2}{8}$. 36. $3x^2-2y^2$.

III. c. (p. 19).

1. $2a$. 2. $5x$. 3. $2a$. 4. $5x+2a$. 5. $2a-b$.
6. $5a-2b$. 7. $2x^2$. 8. $5x^2-3y^2$. 9. a. 10. $a+b$.
11. $a+b$. 12. $a+\frac{b}{3}$. 13. $a-c$. 14. $a+b-2c$.
15. $3a-3b-3c$. 16. $2x^2+6x+4$. 17. $3x^2-3x-3$. 18. x^3-x^2-x.
19. x^2+2. 20. $3x^2+x-5$. 21. $2a$. 22. $6a-3c$.
23. $4x-y+3z$. 24. b^2. 25. $5x^2+3x$. 26. $2x^2+2y^2$.
27. $5(a-b)$. 28. $a+b$. 29. x^2-y^2. 30. $x+5$.
31. $a-b$. 32. $-(x-3)$. 33. $8\frac{1}{2}$. 34. $3\frac{3}{4}$.
35. 6. 36. 7. 37. $6a-2b$. 38. $x+5y$.
39. $10x-15$. 40. $9-5x$. 41. $9+2x$. 42. $2ax$.

III. d. (p. 20).

1. $14a$. 2. $2a$. 3. $-10x$. 4. $9x^2$.
5. $-3y$. 6. 0. 7. $5ab$. 8. 0.
9. $-3x^3$. 10. $2x$. 11. $4a$. 12. $\frac{4x}{y}$.
13. $\frac{5x}{4}$. 14. $2x$. 15. $4a$. 16. $5x^2$.
17. $4ab$. 18. $4x^2y$. 19. $-6abc$. 20. $-15x^4$.
21. 0. 22. $\frac{2x}{3}$. 23. $-\frac{x}{9}$. 24. $-4a^2$.

III. e. (p. 21).

1. $a^2-b^2+c^2$. 2. $6a+6b+6c$. 3. $2x-y-9z$.
4. $-6a-6b-6c$. 5. $13ax+3by+4cz$. 6. $2a+2b+2c$.
7. $4a$. 8. $8a-6b-2c$. 9. $2x^2+4xy+y^2$.
10. $3x^3+y^3$. 11. x^3+3x^2+8x+7. 12. $4a^3-2b^3-5c^3+3d^3$.

13. $2x^3-5x^2y+2xy^2+3y^3$. 14. p^2-3q^2.
15. $5x^2yz-6xy^2z-6xyz^2$. 16. $a^2+b^2+ab-4bc-3ac$.
17. $a^3+4a^2c+3abc+ac^2$. 18. $2a+9b+17c$.
19. $-\frac{2x}{3}+\frac{4y}{3}+\frac{2z}{3}$. 20. $a+2b+5c$. 21. $12x-10y$.

III. f. (p. 22).

1. $3a$. 2. $5a$. 3. $-5a$. 4. $7b$. 5. $-5b$.
6. 0. 7. $19b$. 8. $-2x$. 9. $4y$.
10. $-2x^3$. 11. $4ax^2$. 12. $-4ax^2$. 13. $18ax^2$.
14. $-20ax^2$. 15. $-a$. 16. $-11a$. 17. $3a$.
18. $-3a-2b$. 19. $-a+b$. 20. $2b$. 21. $a-2b$.
22. b. 23. $\frac{a}{2}+\frac{b}{2}$. 24. $\frac{a}{2}-\frac{b}{2}$. 25. $a+b-c$.
26. $c-a-b$. 27. $ax-a$. 28. $ax+a$. 29. $a-ax$.
30. x^2-x. 31. b. 32. $-3b$. 33. $c-b$.
34. $2b+c$. 35. $4y^2-x^2+2z^2$. 36. $12+10x-x^2$. 37. $2x^2-2px-q$.

III. g. (p. 23).

1. $2b^2$. 2. $4x+4y-5z$. 3. $2x^2-2x+4$.
4. $-2x^2+4xy+8y^2$. 5. $-a-2b+c+4d$. 6. $2x-4a-13$.
7. $8b^2+8ab-9$. 8. $a-2b-6d$. 9. $-3x^2y-2xy^2+y^3$.
10. $3a-2b+2c-2d$. 11. $x-5y-z-2$. 12. $5a^2-4ab-14$.
13. $4x^3+9x^2+5x-17$. 14. a^3-9a^2+6a+6.
15. $2ab-2bc+2cd-ad$. 16. $2a^4+2a^3-5a^2-3a+1$.
17. $6x^4-3x^3-6x-29$. 18. 3. 19. 11. 20. 2.
21. $4a$. 22. x^2. 23. x^2-4x. 24. $2b$. 25. $2a-11x$. 26. $8a$.
27. $7a-5$. 28. $3x^2+x$. 29. 6. 30. $a+5b$. 31. 7.
32. $13\frac{1}{3}$. 33. $2a-8b$. 34. $-2x+5y-z$. 35. $6-7x$.
36. $-3a^2+b^2-c^2$. 37. $a+b+d$. 38. $-x^2-3x$.

IV. a. (p. 26).

1. $6a$. 2. $-9a$. 3. $8a$. 4. $2a^3$.
5. $-2a^4$. 6. $-6a^2b^2$. 7. $12xy$. 8. $6xy$.
9. $-15xy$. 10. $-14x^3$. 11. $a^2b^2c^2$. 12. $-a^2b^3c$.
13. $-a^2x^3$. 14. $6a^3b$. 15. $-8x^5$. 16. $-p^{14}$.
17. p^8q^8. 18. $-6p^3q^4$. 19. $a^3b^5c^7$. 20. $\frac{ab}{6}$.
21. $-a^3b^2$. 22. $-\frac{5x^4}{3}$. 23. $\frac{x^2y^3z}{2}$. 24. $-\frac{9a^2b^2c^2}{5}$.
25. 24. 26. $-abc$. 27. $-a^2b^2c$. 28. ab^2c^2.
29. $30abc$. 30. $24abc$. 31. $-a^2x^2y$. 32. $-3ax^3$.
33. $-a^8$. 34. $-8a^3$. 35. $2a^2b^3c^4$. 36. $24p^3q^2r$.

37. a^2. **38.** $-a^3$. **39.** a^6. **40.** $-8a^3$. **41.** x^6.
42. x^6. **43.** $-x^6$. **44.** $-8x^3y^3$. **45.** $16x^4y^4$. **46.** -1.
47. 1. **48.** -1. **49.** $-x^{14}$. **50.** $-x^{15}$.
51. $64x^{12}$. **52.** $-8a^6b^3$. **53.** $-27x^6y^3$. **54.** $81x^4y^8$.

IV. b. (p. 27).

1. $5a+25b-15c$. **2.** $-8a+12b-8c$. **3.** $2a^2+2ab+2ac$.
4. $-6a^3+4a^2-10a$. **5.** $42a^5-28a^4-14a^3-35a^2$.
6. $ab^2c-b^2c^2+abc^2$. **7.** $-6a^2b^2c+9ab^2c^2+12a^2bc^2$.
8. $x^5-2x^4y+x^3y^2$. **9.** $-3x^5+9x^4y-9x^3y^2+3x^2y^3$.
10. $-a^2c-abc-b^2c+ac^2+bc^2$. **11.** $3a^2b^2c+2a^2bc^2-ab^2c^2$.
12. $-2x+6x^2+4x^3-2x^4$. **13.** $2x^4-6x^3+6x^2+2x$.
14. $-15x^6+10x^4-30x^2$. **15.** $6a^2b^2+4ab^3-2b^4$.
16. $60a^9b^4c^3+12a^7b^8c^6-108a^6b^9c^5$. **17.** a^2b-ab^2.
18. $6a^3c-12a^2bc-6ab^2c$. **19.** $-6x^4+30x^3-18x^2$.
20. $12x^6-36x^5+24x^4-36x^2$. **21.** a^{m+n}. **22.** $-a^{m+n}$.
23. a^{2m}. **24.** a^{3m}. **25.** a^{n+3}. **26.** $-a^{n+5}$. **27.** a^{5m}.
28. a^{2m+2n}. **29.** $-2a^{2m}$. **30.** $15a^{m+n}b^{m+n}$. **31.** $a^{2x}+a^{3x}$.
32. $e^{4x}-e^{3x}+e^{2x}$. **33.** a^{2m}. **34.** a^{2m-8}.
35. 2. **36.** 14. **37.** 0. **38.** -8. **39.** 0. **40.** 2.
41. 3. **42.** -7. **43.** 5. **44.** 0. **45.** 3. **46.** 7.
47. 5. **48.** -2. **49.** 7. **50.** 7. **51.** 17. **52.** 14.

IV. c. (p. 29).

1. x^2+5x+6. **2.** x^2-5x+6. **3.** x^2-x-6.
4. x^2+x-6. **5.** $x^2+12x+27$. **6.** $x^2+3x-18$.
7. $x^2-18x+77$. **8.** $x^2+4x-77$. **9.** $1+3x+2x^2$.
10. $1+x-12x^2$. **11.** $1-3x+2x^2$. **12.** $6+5x+x^2$.
13. $30+11x+x^2$. **14.** $21+10x+x^2$. **15.** $1-2x-63x^2$.
16. $1-4x-21x^2$. **17.** x^2-1. **18.** x^2-4.
19. x^2-9. **20.** x^2-49. **21.** $1-x^2$.
22. $4-x^2$. **23** $49-x^2$. **24.** $81-x^2$.
25. $x^2+2xy+y^2$. **26.** $x^2+5xy+6y^2$. **27.** x^2-4y^2.
28. $x^2-5xy+6y^2$. **29.** $x^2-xy-6y^2$. **30.** $x^2-xy-20y^2$.
31. $4x^2+4xy+y^2$. **32.** $9x^2-6xy+y^2$. **33.** $6x^2-x-12$.
34. $6x^2-11x+4$. **35.** $10x^2+27x+18$. **36.** $15x^2-29x-14$.
37. $6-13x+6x^2$. **38.** $30+11x-28x^2$. **39.** $4-9x^2$.
40. $4x^2-25$. **41.** $25x^2-49$. **42.** $36x^2-25$.
43. $81x^2-64$. **44.** $16x^2-49$. **45.** $x^2-ax+bx-ab$.
46. $x^2+ax-bx-ab$. **47.** $a^2+2ab+b^2$. **48.** $a^2x^2+2abx+b^2$.
49. $a^2-2ab+b^2$. **50.** $a^2x^2-2abx+b^2$. **51.** $p^2x^2-2pqx+q^2$.
52. $p^2+2pqx+q^2x^2$. **53.** $a^2-2ax-15x^2$. **54.** $21-x-2x^2$.

55. $x^2 - a^2y^2$. **56.** $p^2x^2 - q^2$. **57.** $p^2x^2 + 2pqx + q^2$

58. $c^2x^2 - 2cdx + d^2$. **59.** $12x^2 - 25xy + 12y^2$. **60.** $12x^2 + xy - 20y^2$

61. $42x^2 + 20cx - 32c^2$. **62.** $6a^2x^2 + 13ax + 6$. **63.** $a^4 - b^4$.

64. $a^4 - 16b^2$. **65.** $a^4 + 2a^2b - 24b^2$. **66.** $a^4 - 8a^2b + 15b^2$.

67 $16a^4 - 9b^2$. **68.** $25a^4 - 4b^4$. **69.** $x^4 - 4a^4$.

70. $x^4 - p^2$. **71.** $a^2 - b^6$. **72.** $a^2 - 2ab^3 + b^6$.

73. $x^6 - 1$. **74.** $x^6 - 4$. **75.** $a^2x^4 - 1$. **76.** $b^2x^4 - c^2$.

77. $abx^2 + ax + bx + 1$. **78.** $abx^2 - ax + bx - 1$.

79. $3x^2 + 6xy + x + 2y$. **80.** $6x^2 - 3ax + 2bx - ab$

81. $ac + bc + ad + bd$. **82.** $ac - bc - ad + bd$.

83. $6ac - 3bc + 8ad - 4bd$. **84.** $2ac + 6bc - 5ad - 15bd$.

85. $x^4 + ax^2 - 3bx^2 - 3ab$. **86.** $a^2x^3 + 2abx^2 + b^2x$.

87. $a^2x^3 - b^2x$. **88.** $x^3 + ax^2 + a^2x + a^3$.

89. $x^3 + ax^2 - a^2x - a^3$. **90.** $x^3 - 2x^2y - 4xy^2 + 8y^3$.

IV. d. (p. 31).

1. $a^2 + 2ab + b^2$. **2.** $a^2 + 2ax + x^2$. **3.** $c^2 + 2cd + d^2$.

4. $x^2 + 8x + 16$. **5.** $x^2 + 14x + 49$. **6.** $p^2 + 6p + 9$.

7. $a^2 - 2ab + b^2$. **8.** $a^2 - 2ax + x^2$. **9.** $c^2 - 2cd + d^2$.

10. $x^2 - 8x + 16$. **11.** $x^2 - 18x + 81$. **12.** $p^2 - 8p + 16$.

13. $4p^2 + 12p + 9$. **14** $9p^2 + 6pq + q^2$. **15.** $4p^2 - 20p + 25$.

16. $16p^2 - 8p + 1$. **17.** $x^2 - 2x + 1$. **18.** $9x^2 - 6x + 1$.

19. $1 - 2x + x^2$. **20.** $1 - 4x + 4x^2$. **21.** $1 - 10x + 25x^2$.

22. $1 + 2p + p^2$. **23.** $1 + 14p + 49p^2$. **24.** $4a^2 + 12ab + 9b^2$.

25. $16x^2 - 24xy + 9y^2$. **26.** $a^2 - 2ab + b^2$. **27.** $4a^2 - 4ax + x^2$.

28. $4x^2 - 12ax + 9a^2$. **29.** $4x^2 - 12ax + 9a^2$. **30.** $16p^2 + 40pq + 25q^2$.

31. $25p^2 - 40pq + 16q^2$. **32.** $a^4 + 2a^2b^2 + b^4$. **33.** $a^4 - 2a^2b^2 + b^4$.

34. $a^4 + 2a^2b + b^2$. **35.** $a^4 - 2a^2p + p^2$. **36.** $4a^4 - 12a^2b^2 + 9b^4$.

37. $16a^4 + 24a^2b^2 + 9b^4$ **38.** $a^6 + 2a^3b + b^2$. **39.** $x^6 + 2x^3y^3 + y^6$.

40. $x^6 - 2x^3y^3 + y^6$. **41.** $4x^4 + 4ax^2 + a^2$. **42.** $9x^4 - 6x^2y^2 + y^4$.

43 $1 - 4x^2 + 4x^4$. **44.** $1 + 2x + x^2$. **45.** $1 + 4x + 4x^2$.

46. $x^8 + 2x^4a^4 + a^8$. **47.** $x^8 - 2x^4y^4 + y^8$. **48.** $4x^8 - 12x^4y^4 + 9y^8$.

49. $4p^6 + 12p^3q^2 + 9q^4$. **50.** $x^{10} - 2x^5a^5 + a^{10}$.

IV. e. (p. 31).

1. $x^2 - 1$. **2.** $x^2 - 4$. **3.** $1 - x^2$. **4.** $x^2 - 25$.

5. $9 - y^2$. **6.** $49 - x^2$. **7.** $b^2 - a^2$. **8.** $4p^2 - q^2$.

9. $9p^2 - q^2$. **10.** $a^2 - 9b^2$. **11.** $9p^2 - 4q^2$. **12** $25x^2 - 16a^2$.

13. $a^2 - b^2$. **14.** $4a^2 - x^2$. **15.** $a^2 - 49b^2$. **16.** $a^2 - 49b^2$.

17. $x^4 - y^4$. **18.** $a^4 - 4b^4$. **19.** $p^2x^2 - q^2$. **20.** $a^2 - b^2x^2$.

21. $x^6 - a^6$. **22.** $x^4 - a^2$. **23.** $4a^6 - x^2$. **24.** $4a^4 - 9x^2$.

25. $1 - x^6$. **26.** $1 - a^2x^4$. **27.** $9 - a^6$. **28.** $121 - 49x^2$.

29. $81 - 64x^2$. **30.** $49x^2 - 81$.

IV. f. (p. 32).

1. 9604. **2.** 40401. **3.** 10404. **4.** 10609.
5. 11449. **6.** 99980001. **7.** 1002001. **8.** 1004004.
9. 98·01. **10.** 100060009. **11.** 400040001.
12. 999600·04. **13.** 400400100. **14.** 4020025.
15. 10060·09. **16.** 1016064. **17.** 998001.
18. 9994·0009. **19.** 6432·04. **20.** 360600·25.
21. 809280·16. **22.** 250300·09. **23.** 81·108036.
24. 63·936016. **25.** 10004·000. **26.** 1·0100.
27. 101·606. **28.** 999920·00. **29.** 100·1000.
30. 999996. **31.** 39991. **32.** 9991. **33.** 6391.
34. 120·75. **35.** 99·51. **36.** 6396. **37.** 399·9984.
38. 2·8896. **39.** 3·9984. **40.** 80999999·84.

IV. g. (p. 33).

1. x^3-3x^2+3x-1. **2.** x^3+5x^2+8x+4. **3.** $4x^3-8x^2+5x-1$.
4. x^3+8. **5.** $27x^3-1$. **6.** $6x^3+11x^2-2x+20$.
7. $x^3-2ax^2+2a^2x-a^3$. **8.** $125x^3-1$. **9.** $a^3+a^2b+ab^2+b^3$.
10. x^3-a^3. **11.** $a^3+a^2b-ab^2-b^3$. **12.** $x^3-9x^2+27x-27$.
13. $8x^3-1$. **14.** $8x^3-32x^2+4x+35$.
15. $4x^3-8x^2-3x+6$. **16.** $x^4+3x^3-6x^2-6x+8$.
17. $27x^3+1$. **18.** x^4+2x^3-2x-1.
19. $x^3-ax^2-bx^2-cx^2+abx+bcx+cax-abc$. **20.** x^4-16a^4.
21. $x^4-18b^2x^2+81b^4$. **22.** $12x^3-16x^2-79x-42$.
23. $a^3-a^2c-ab^2+b^2c$. **24.** $a^2-b^2-ac+bc$.
25. $6a^2+ab-3ac+4bc-12b^2$.

IV. h. (p. 33).

1. 9. **2.** 4. **3.** -5. **4.** 17. **5.** 1.
6. -13. **7.** $x+3$. **8.** $3x-6$. **9.** $6x-10$. **10.** $-3x$.
11. 5. **12.** 11. **13.** 0. **14.** $6-a$. **15.** 0.
16. -31. **17.** $ad+b$. **18.** 0. **19.** 6. **20.** 31.
21. c^2+b^2. **22.** 0. **23.** $a^2+2ab+b^2$.
24. $21x^3+8x^2-39x+10$. **25.** x^2-6x. **26.** 42.
27. $20x^2-5ax$. **28.** $16x^2-8x$. **29.** $26x-10$. **30.** $16p-4q$.
31. $9x^3-6x^2+7x-2$. **32.** $2a^2+5ab+2b^2$. **33.** 7. **35.** $14x$.

V. a. (p. 36).

1. x. **2.** 3. **3.** x. **4.** $-x$. **5.** bc.
6. $-bc$. **7.** a. **8.** $-a$. **9.** $-x$. **10.** x.
11. a^3. **12.** $-a$. **13.** 1. **14.** 1.

15. $4x^2$. 16. $-3x^2$. 17. -2. 18. $3a^2$.
19. $-7a^2x^3$. 20. a^2b^5. 21. $-9a$. 22. $4abc$.
23. $-3x^3$. 24. $-9ab^2c^5$. 25. $3a$. 26. 6.
27. $-6a$. 28. $8a$. 29. $-6ab^2$. 30. xyz^2.
31. $24a^6b^4$. 32. $3p^3q^2x$. 33. $-7a^3c^4$. 34. $-7qr$.
35. $-8ln$. 36. $-9a^2b^4c^6$. 37. $-18ax^4$. 38. $11xy^5$.

V. b. (p. 36).

1. $a-2b$. 2. $-a+3b$. 3. $4x-3$. 4. $-y+6$.
5. $a+b$. 6. $b-a$. 7. $a-2b$. 8. $a-3b$.
9. $-3a^2+7b^2$. 10. $b+c$. 11. $-a-b$. 12. $4x-5$.
13. $7x-9$. 14. a^2b-ab^2. 15. $3a-7b$. 16. $6x^4y^5z-5x^2y^3z^4$.
17. $-2a+b$. 18. $11x+6y$. 19. $2a^2-4b^2$. 20. m^2-4mn.
21. $-4a+3b+6c$. 22. $a+c+d$. 23. $-3a+4d+12x$.
24. $-a-x-ax$. 25. $-a+4b-8c$. 26. x^2+3x-3.
27. $-x^2+ax-a^2$. 28. $a+5b^2-3b$. 29. $-a+b-c$.
30. $-2x^3+x^2-4x+1$. 31. $3y^3-xy^2-6x^3$. 32. $-3xy+7y^2+x^2$.
33. $-xy^5+2x^2y^3+7x^3y$. 34. $3xy^2z^4-5x^2yz^3+6x^3y^4z^2$.
35. a^{m-n}. 36. a^{n-3}. 37. x^{4-p}. 38. $-3x^{n-4}$.
39. $9x^{m-n}y^{n-m}$. 40. $9x^{3-n}y^{3-n}$.

V. c. (p. 38).

1. $x+4$. 2. $x-4$. 3. $a+1$. 4. $a-1$.
5. $b+7$. 6. $x+3$. 7. $x-7$. 8. $x-1$.
9. $a-6$. 10. $y+9$. 11. $x-2$. 12. $5x+3$.
13. $2x-1$. 14. $3x-7$. 15. $3x+1$. 16. $2x-4$.
17. $2+x$. 18. $1-2x$. 19. $3-x$. 20. $a-2$.
21. $5-3a$. 22. $5y+11$. 23. $x-a$. 24. $5x+4$.
25. $a+2x$. 26. $5-x$. 27. $1+2x$. 28. $x+2y$.
29. $1+8pq$. 30. $3a-b$. 31. $a-bc$. 32. $2x^2+7$.
33. $9x^3-1$. 34. $5x^2+4y^2$. 35. $10-x$. 36. $1+10b^2$.

V. d. (p. 39).

1. x^2+a^2. 2. $x+b$. 3. $x-a$. 4. $x+1$.
5. $x+a$. 6. $x-2$. 7. $px+1$. 8. $x+1$.
9. $x-a$. 10. $px+2$. 11. $ax-5c$. 12. $ax+c$.
13. $x-7$. 14. $ax+b$. 15. $3ax+2b$. 16. $ax-b$.
17. $9x+bc$. 18. $2x+bq$. 19. $bx+c$. 20. $5px+3q$.
21. $x-3$. 22. $15(x-3a)$. 23. $2x+3$. 24. x^2+2x+1.
25. $21(x+3)$. 26. $2x^3-11x^2+4x+5$, x^2-6x+5. 27. $2x-3$.
28. a^3-a^2b [illegible] $+b^3$. 29. 18. 30. 33. 31. $x-1$.
32. $2x-1$. 33. -4. 34. $bx-c$. 35. $ax+b$. 36. $a+2b$.

VI. a. *Oral.* (p. 40).

1. (i) x. (ii) $\frac{3x}{2}$. (iii) $\frac{x}{2}$. (iv) $\frac{9ab}{2}$. (v) $\frac{5abc}{2}$. (vi) $\frac{5a}{2}$.

2. (i) 9. (ii) -11. (iii) 0. (iv) 1. (v) -5. (vi) $\cdot 5$.

3. (i) 25. (ii) 9. (iii) $\frac{1}{9}$. (iv) $\frac{1}{4}$. (v) 1. (vi)
(vii) $\frac{a^2b^2}{4}$. (viii) $-\frac{a^3b^3}{8}$.

4. (i) 13. (ii) 25. (iii) -5. (iv) 1. (v) 9. (vi) 27.

5. (i) 5. (ii) $-a$. (iii) $-3a$. (iv) $7x^2$. (v) 0. (vi) 3.

6. (i) 0. (ii) 3. (iii) $-\frac{3}{4}$. (iv) 8. (v) $1\frac{1}{4}$. (vi) $5\frac{1}{4}$.

7. (i) 7. (ii) 3. (iii) 13. (iv) 1. (v) 1. (vi) 31.

8. (i) -1. (ii) 0. (iii) -6. (iv) 3. (v) 14. (vi) -52.

9. (i) $5x$. (ii) $5a$. (iii) $3x^2$. (iv) $2ab$. (v) $9x-20$. (vi) 2.

10. (i) 2. (ii) x. (iii) $x+2$. (iv) $x-1$. (v) $x-1$. (vi) $x+2$.
(vii) $4x+2$. (viii) $a+b+c$.

11. (i) bx^2. (ii) $-2cx$. (iii) x^3.

12. (i) $a-b$. (ii) $c-b$.

13. (i) $-4a$. (ii) $4a$. (iii) 0. (iv) $\frac{5x}{4}$. (v) x^2-1. (vi) x^2+1.
(vii) x^3+1. (viii) x^2-5x+1. (ix) $7(x-1)$. (x) $x-3$. (xi) $a+bx$.
(xii) a. (xiii) $4bx$. (xiv) 2.

14. (i) $4x-3y+z$. (ii) $3x^2$. (iii) $a+5b+3c$. (iv) $x^3-x^2y+xy^2$.
(v) $4x^3-4x^2-5$. (vi) $2a-b$.

15. (i) $4x$. (ii) x^2+xy. (iii) $\frac{x}{4}$. (iv) $3y-2x$. (v) $2a^2x$. (vi) $8b$.
(vii) 0. (viii) $2a-2b$. (ix) $4(2-x)$. (x) $a+b$. (xi) $2b-2a$.
(xii) $2x-6$. (xiii) x^3-x^2. (xiv) $5x^3-8x^2+5x+1$. (xv) $2(x-y)$.
(xvi) $2(b-2a)$. (xvii) $3x^2$. (xviii) $2bc$. (xix) $2(x-y+z)$.

16. (i) $4x$. (ii) $7x^2-4$. (iii) $-x^2$. (iv) $2a^2x$. (v) $6-2x^2$.
(vi) $4(-b)$. (vii) x^3-14x^2+5. (viii) $-7(a^2-b^2)$. (ix) 141.
(x) 5. (xi) 81. (xii) 24.

17. (i) $-6ab$. (ii) -1. (iii) $-ax$. (iv) $\frac{a}{2}$. (v) $3a^3b^3c^3$. (vi) $3b$.
(vii) $-\frac{3x^7}{2}$. (viii) $9x^2$. (ix) $-\frac{a^3x}{9}$. (x) $\frac{3x}{2}$. (xi) a^2x^2. (xii) $-ax$.
(xiii) $-a^7$. (xiv) $-a$. (xv) $-a^{10}$. (xvi) -1.

18. (i) $4ax^2y-3axy^2$. (ii) $-2x^3+6x^2-x$. (iii) $3x^2+4x-2$.
(iv) $4x^2-2x+3$. (v) $-3x^2+2x+9$. (vi) $-18x^4+12x^3-6x^2$.

19. (i) $1-x^2$. (ii) $1+2x+x^2$. (iii) $1-4x+4x^2$. (iv) $a^2+4ab+4b^2$.
(v) $x^2+8x+15$. (vi) x^2-x-6. (vii) $x^2-5xy+6y^2$. (viii) $9x^2-1$.
(ix) $30-11p+p^2$. (x) a^4-9. (xi) $9x^2-25$. (xii) $a^4x^2+2a^2x+1$.
(xiii) $2x^2-32$. (xiv) $x^4+5x^2y+6y^2$. (xv) $1+2x-8x^2$. (xvi) a^2-4b^2.
(xvii) $1+4x+4x^2$. (xviii) $3a^2-3x^2$. (xix) $4a^2-1$. (xx) $9x^4-1$.

20. (i) $9a^2-12ab+4b^2$. (ii) $4a^2-4ay+y^2$. (iii) a^4-4a^2+4.
(iv) $x^2+ax+\frac{a^2}{4}$. (v) $4x^2-4x+1$. (vi) $9x^2-6x+1$.
(vii) $21+4x-x^2$. (viii) $75-3x^2$. (ix) $2x^2-4xy+2y^2$.
(x) $x^2+cx-ax-ac$. (xi) x^2-5x+6. (xii) $x^2-\frac{4}{9}$.
(xiii) $a^2+2ax-8x^2$. (xiv) $abx^2-ax-bx+1$. (xv) $9a^2-\frac{1}{4}$.
(xvi) $36x^2-1$. (xvii) $10x^2+9x-9$. (xviii) $15x^2+29x-14$.
(xix) $15x^2+13x+2$. (xx) $14x^2+xy-3y^2$.

21. (i) 3. (ii) -7. (iii) -27. (iv) -2.

22. (i) 3. (ii) 11. (iii) 16. (iv) -8.

23. (i) $-x^3$. (ii) $2a^2$. (iii) $7a$. (iv) $\frac{5a}{3}$. (v) $4r$.
(vi) $-\frac{27p^3q}{4}$. (vii) $3b-4a$. (viii) $3x^2+1$. (ix) $3a-4x$. (x) $3b-4a$.
(xi) $-a^2+bc-c^2$. (xii) $-x$. (xiii) $a-x$. (xiv) $2(a-b)$. (xv) x.
(xvi) $5a$. (xvii) $(a+x)^2$. (xviii) ax. (xix) 2. (xx) $(a-x)^2$.

VI. b. (p. 44).

1. 0, 1, 9. **2.** $2x^4-7x^3+5x-3$, -7, 0.
4. $a+4b$, $6b$. **5.** $6x^2+7ax-20a^2$, ax^2-a^3.
6. $7x$, $-3x^2$, $2a^2-3ab+4b^2$. **7.** $3x-y$.

VI. c. (p. 44).

1. 0, 9, 1. **2.** $b^4+2ab^3+5a^2b^2-3a^3b+a^4$, $-3b$.
4. x^2-1. **5.** x^2-9a^2, $x^4-ax^3-2a^2x^2$.
6. (1) x, (2) $x-3a$, (3) a^2bc. **7.** $3a+4b$.

VI. d. (p. 45).

1. 6, 12, 0. **2.** x^3-3x^2+3x-1, 0. **3.** 0, $x-8$.
4. $a-13c+6b$. **5.** $-a^3b^2$, a^7x^5, $-a^4b^4c^4$.
6. $6x-9a$. **7.** $21p^2+pq-36q^2$.

VI. e. (p. 45).

1. 1, -1, 64. **3.** $10a+2x$, $x^3+3x^2-16x-4$, 32.
4. x^2-2x+3. **5.** $15x^2+3ax$.
6. $ax-8a$. **7.** $-6y^2$.

VI. f. (p. 45).

1. -1, 0, 0. **3.** $x-7$, $2(x-1)$.
4. $3x^3-4x^2+6x-2$, 18. **5.** $7x^2-17ax-12a^2$.
6. $18x^4+9ax^2-2a^2$. **7.** $5x-4a$.

VI. g. (p. 46).

1. $-3, -20$.
2. 2 miles East.
3. $x^2-2ax+a^2, \ ax-2x^2$.
4. $11ax^2$.
5. $2a^2-5ab+3b^2$.
6. $4x^2-a^2, \ x^4-9, \ a^2-p^4$.

VI. h. (p. 46).

1. 27, 44.
2. $6x^2+2$.
3. $5x-y-6a$.
4. $5x^2+10$.
5. 15, 1, -3, 3, 19.
6. $x^3-2x^2y-4xy^2+8y^3$.
7. $ax+3p$.

VI. k. (p. 46).

1. $-33, -25$.
2. $2x^2-3x$.
3. $a^2-b^2-c^2+2bc$.
4. $2x, \ 2y$.
5. 23, 9, 1, -1, 3.
6. $x^3+ax^2-a^2x-a^3$.
7. $4x-5$.

VII. a. (p. 48).

1. 3. 2. 3. 3. 4. 4. -5. 5. 3. 6. -3.
7. -6. 8. 0. 9. 5. 10. 2. 11. 12. 12. -8.
13. -20. 14. 0. 15. $2\frac{1}{2}$. 16. $2\frac{1}{4}$. 17. 9. 18. 2.
19. 1. 20. $\frac{2}{3}$. 21. $\frac{1}{2}$. 22. $-\frac{1}{4}$. 23. 8. 24. -15.
25. 0. 26. $1\frac{1}{2}$. 27. 3. 28. -3. 29. 1. 30. 0.
31. -1. 32. -1. 33. 2. 34. 4. 35. 2. 36. 2.
37. 3. 38. $3\frac{1}{2}$. 39. 2. 40. 2. 41. 20. 42. 3.
43. 3. 44. ·01. 45. ·03. 46. $-$·03.

VII. b. (p. 49).

1. 2. 2. 12. 3. 7. 4. 1. 5. 3.
6. 12. 7. $1\frac{1}{3}$. 8. -3. 9. 0. 10. 0.
11. 2. 12. 2. 13. 5. 14. -3. 15. 5.
16. 4. 17. -1. 18. 0. 19. 3. 20. -2.
21. $1\frac{1}{2}$. 22. $1\frac{1}{4}$. 23. -27. 24. $1\frac{1}{2}$. 25. -6.
26. $1\frac{1}{20}$. 27. -9. 28. $-8\frac{1}{2}$. 29. 0. 30. 3.
31. $2\frac{1}{2}$. 32. 2. 33. 5. 34. 5. 35. 3.
36. $1\frac{1}{4}$. 37. 3. 38. -5. 39. 10. 40. $3\frac{1}{2}$.
41. $-2\frac{1}{3}$. 42. $2\frac{1}{2}$. 43. 0. 44. 2.

VII. c. (p. 51).

1. 3. 2. 10. 3. 14. 4. 22. 5. 3. 6. 28.
7. -11. 8. -3. 9. 7. 10. 28. 11. 2. 12. 3.
13. 9. 14. -20. 15. 1. 16. 2. 17. [illegible] 18. -5.
19. $-1\frac{1}{3}$. 20. $\frac{1}{2}$. 21. -2. 22. 3.
23. 1. 24. 7. 25. 5. 26. 9.

VII. d. (p. 53).

1. 18. 2. 12. 3. 15. 4. 4. 5. 70. 6. 12.
7. 4. 8. 14. 9. 19. 10. 7. 11. 2. 12. $3\frac{1}{2}$.
13. 5. 14. 2. 15. -1. 16. 1. 17. 4. 18. 11.
19. 7. 20. 11. 21. -6. 22. $-1\frac{1}{5}$. 23. 12. 24. 8.
25. 4. 26. 8. 27. 12. 28. 12. 29. -7. 30. 8.
31. 2. 32. 10. 33. -1. 34. $-\frac{1}{2}$. 35. $1\frac{2}{3}$. 36. -7.
37. 0. 38. -2. 39. 2. 40. 2. 41. 15. 42. 17.
43. $-\frac{16}{17}$. 44. 9. 45. 2. 46. 3. 47. 2. 48. 7.
49. $\frac{1}{3}$. 50. 3. 51. 1. 52. 5. 53. 14. 54. 14.
55. 7. 56. $30\frac{19}{29}$. 57. 3. 58. 15·5. 59. 1. 60. 1·5.
61. 140. 62. ·69. 63. 3. 64. 1·95.
65. 2. 66. 1·1. 67. 1. 68. $1\frac{1}{17}$.
69. When $x=-4\frac{3}{5}$. 70. 1. The equation has no root. 71. No root.

VII. e. (p. 55).

1. 10. 2. 4·7. 3. $-·78$. 4. 4·33. 5. 5·71. 6. 26.
7. 2·53. 8. 46·83. 9. $-1·43$. 10. ·46. 11. 3·03. 12. 2·04.

VIII. a. (p. 57).

1. $x-20$. 2. $35-y$. 3. $x-20$. 4. $34-x$. 5. $\frac{56}{x}$.
6. $35x$. 7. 21. 8. $x-23$. 9. $y-x$. 10. $x-13$.
11. $\frac{78}{x}$. 12. $\frac{x}{y}$. 13. $\frac{5b}{3a}$. 14. $20x-y$. 15. $96-x-y$.
16 $a+2b$. 17. $2y-x$. 18. $\frac{y}{x}$. 19. $\frac{12y}{x}$. 20. $\frac{12x}{y}$.
21. $20y-\frac{5x}{2}$. 22. $x+4$. 23 $4+x$. 24. $20-x$. 25. $40-a$.
26. 25. 27. $\frac{4}{x}$ pence. 28. $x+7$, $x+y$, $x-11$ years old.
29. $\frac{3x}{2}$. 30. $\frac{x}{6}$ miles, $\frac{xy}{6}$ miles, $\frac{6}{x}$ hours, $\frac{6y}{x}$ hours. 31. $2b$.
32. $\frac{x}{y}$. 33. $3x$ pence. 34. $\frac{x}{4}$ pence. 35. 2.
36. $\frac{x}{12}$ pence, $\frac{144}{x}$ eggs, $\frac{144y}{x}$ eggs. 37. $\frac{8x}{3}$ pence. 38. $\frac{yz}{x}$ pence.
39. n, $n+1$, $n+2$. 40. $n-2$, $n-1$, n. 41. $n-1$, n, $n+1$.
42. n, $n+1$, $n+2$. 43. $n-2$, $n-1$, n, $n+1$, $n+2$. 44. $\frac{xy}{20}$.
45. 2x[illegible] 46. $4b$. 47. $240a+12b+c$.
48. $\frac{88x}{[illegible]}$. 49. $10x$ miles. 50. $\frac{532}{x}$ days, $\frac{532}{xy}$ days.

51. $\frac{x}{4}+25$. **52.** $2n-1,\ 2n-2,\ 2n-3,\ 2n-4,\ 2n-5$.

53. $2n-5,\ 2n-3,\ 2n-1,\ 2n+1,\ 2n+3$. **54.** ab sq. ft. **55.** $\frac{x}{y}$ feet.

56. x^2 sq. ft. **57.** $x-20=y$. **58.** $3x-y=25$.

59. $\frac{x-8}{6}=\frac{2x+3}{7}$. **60.** $3(x-4)=5(x-1)$. **61.** $20y+2z=x$.

62. $240b+30c+12d=a$. **63.** $x(x-1)=y$. **64.** $(x-1)x(x+1)=a^3$.

65. $2x+5=y$. **66.** $2x-y=a$. **67.** $x+a=y-a$.

68. $x=15y+7$. **69.** $a=bx+y$. **70.** $xy=a$.

71. $ab=9x$. **72.** $xy=3(a-b)$. **73.** $x-y=5(a-b)$.

VIII. b. (p. 61).

1. 3 ft. 8 in. **2.** 4 ft. $8\frac{4}{7}$ in. **3.** $17\frac{1}{2}$ ft. **4.** $1\frac{10}{11}$ ft.
5. 31·4 in. **6.** 2·5 in. **7.** 3·2 in. **8.** 50·3 sq. in.
9. 7 in. **10.** 186 sq. ft. **11.** 22 ft. **12.** 12 ft. 5 in.
13. 560 sq. ft. **14.** 12 ft. 6 in. **15.** 10 ft. 10 in. **16.** 198 cub. ft.
17. $4\frac{1}{2}$ ft. **18.** $3\frac{3}{8}$ sq. ft. **19.** 7 ft. 2 in. **20.** 576 ft.
21. 3 secs. **22.** 31 ft. per sec. **23.** 41. **24.** 68. **25.** 325.
26. 460. **27.** 264. **28.** 336. **29.** 1500. **30.** 1892. **31.** 441.
32. 644. **33.** 1625. **34.** 612. **35.** 693. **36.** 1240. **37.** 3220.
38. 13035. **39.** 113·14. **40.** $10\frac{1}{2}$ ft. **41.** £200. **42.** 334.
43. Right-angled. **44.** Not right-angled. **45.** and **46.** Right-angled.
47. Not right-angled. **48.** Right-angled.

VIII. b_1. (p. 63A).

1. (i) 13, (ii) 5, (iii) 1, (iv) 3. **2.** (i) 12, (ii) 17. **3.** a^2-b^2. **4.** -1.
7. (i) 11, (ii) 24. **8.** (i) $9x^2-25$, (ii) 875. **9.** When $x=3$. **10.** $-3\frac{2}{3}$.

VIII. c. (p. 64).

1. 1, 3, 7. **2.** 15, 28, 3, $\frac{(n+1)(n+2)}{2}$, $\frac{(n-3)(n-2)}{2}$.
3. -6, 0, 0, 24, -60.
4. 0, 33, $16n^2-2n$, $16n^2+14n+3$, $4n^2+7n+3$, $\frac{1}{2}$. **6.** 2, 2, 14.
7. x^2+5x+4. **8.** $2b(x+1)$. **9.** $c-a$, $2b$, $3a+4b-3c$, $8a+5b-3c$.
10. $\phi(x-1)=x^3$. **11.** (i) $(3x^2-2x+2)$ miles, (ii) $(3x^2+2x+6)$ miles.
12. (i) $\phi(3)-\phi(2)$, (ii) 80 feet.

IX. a. (p. 66).

1. £10, £20. **2.** 10. **3.** 27. **4.** £15, £25. **5.** 20. **6.** 21.
7. 10 miles. **8.** 3. **9.** 12. **10.** 38, 10 years old. **11.** 36. **12.** 20.
13. £48, £58, £38. **14.** 30, 12. **15.** 20. **16.** 90. **17.** 70 gallons.
18. 31, 32, 33. **19.** 9. **20.** 18 pennies, 9 shill[illegible] 3 florins.
21. £42, £7. **22.** £19, £22. **23.** £336, £164. [illegible] £8. 8s.
25. 45, 20. **26.** 63, 40. **27.** 63, 21. **28.** 72, 12. **29.** 57.

30. -4. **31.** 20. **32.** £420. **33.** 34, 35, 36.
34. 43, 45, 47. **35.** 38 shillings, 19 shillings. **36.** 2 miles.
37. £23. 5s., £16. 15s. **38.** £3600, £720.
39. £13. 10s., £22. 10s. **40.** 15, 42.
41. 29 men, 46 women, 76 children. **42.** 56.
43. $4\frac{1}{2}$ miles an hour, 3 miles an hour.
44. 36 miles an hour, 24 miles an hour. **45.** 150 yards a minute.
46. 24 miles. **47.** $44\frac{1}{2}$ miles. **48.** 30 miles.
49. 36 miles. **50.** 15 miles an hour.

IX. b. (p. 73).

1. 12 miles, nearly. **2.** 13 miles, nearly. **3.** 17 miles, nearly.
4. 3.7 miles an hour. **6.** 5 feet. **7.** 36·1 feet.
8. 2·39 feet. **9.** 4·6 miles. **10.** 35·4 miles. **11.** 4·5 miles.
12. 4·1 miles. **13.** 6·55 metres. **14.** 3·9 in. **15.** 4·24 in.
16. 3·6 feet. **17.** 2·6 in. **18.** 2·2 in. **19.** 3·3 in.
20. 6·4 miles. **21.** 2·83 miles. **22.** 8·05.
23. 15·98. **24.** 3·7 miles. **25.** 14, 29, 43 miles.
26. 2·6 miles. **27.** 34 feet. **28.** 2·8 miles.

IX. c. (p. 77).

1. £24, £35. **2.** 15·1 millions, 1875. **3.** 67·1°.
6. 3 oz. **8.** 4475 feet nearly, 205°.
9. 26 8 in., 23·4 in., 10,600 ft., 5,300 ft.
10. 107·5 sq. in., 162·9 sq. ft., 13·2 in.

X. a. (p. 81).

1. $2y=4$. **2.** $11y=22$. **3.** $4y=12$. **4.** $21y=-13$.
5. $3y=14$. **6.** $y=46$. **7.** $17y=17$. **8.** $58y=87$.
9. $3y=-11$. **10.** $3y=17$. **11.** 2. **12.** 5.
13. 3. **14.** 4. **15.** 11. **16.** $2\frac{1}{2}$.
17. $x=8$, $y=2$. **18.** $x=9$, $y=1$. **19.** $x=2$, $y=1$.
20. $x=1$, $y=2$. **21.** $x=3$, $y=2$. **22.** $x=4$, $y=-1$.
23. $x=-3$, $y=-5$. **24.** $x=2\frac{1}{4}$, $y=\frac{3}{4}$. **25.** $x=4\frac{1}{2}$, $y=0$.
26. $x=15$, $y=1$. **27.** $x=5$, $y=6$. **28.** $x=8$, $y=6$.
29. $x=0$, $y=2$. **30.** $x=4$, $y=0$. **31.** $x=1$, $y=6$.
32. $x=5$, $y=-2$. **33.** $x=1\frac{1}{2}$, $y=\frac{1}{2}$. **34.** $x=13$, $y=7$.
35. $x=1$[illegible], $y=-2\frac{1}{2}$. **36.** $x=3\frac{1}{2}$, $y=2\frac{1}{3}$. **37.** $x=5$, $y=3\frac{3}{5}$.
38. $x=\frac{1}{2}$, $y=1\frac{1}{2}$. **39.** $x=2$, $y=3$. **40.** $x=1$, $y=-1$.
41. $x=7$, $y=5$. **42.** $x=6$, $y=8$. **43.** $x=\frac{1}{2}$, $y=-\frac{1}{2}$.
44. $x=16$, $y=$ [illegible]24. **45.** $x=$ [illegible]6, $y=$ [illegible]. **46.** $x=2$, $y=-1$.

X. b. (p. 83).

1. $x=12,\ y=20.$ 2. $x=20,\ y=12.$ 3. $x=18,\ y=48.$
4. $x=40,\ y=-20.$ 5. $x=-20,\ y=6$ 6. $x=-20,\ y=-40.$
7. $x=2,\ y=3.$ 8. $x=-2,\ y=-3.$ 9. $x=-11\frac{1}{5},\ y=\frac{4}{5}.$
10. $x=45,\ y=10.$ 11. $x=7,\ y=10.$ 12. $x=5,\ y=2.$
13. $x=11,\ y=1.$ 14. $x=3,\ y=6.$ 15. $x=2,\ y=1.$
16. $x=7,\ y=2.$ 17. $x=8\frac{4}{5},\ y=-11.$ 18. $x=13,\ y=9\frac{1}{3}.$
19. $x=48,\ y=7.$ 20. $x=3,\ y=2.$ 21. $x=10,\ y=2.$
22. $x=\cdot3,\ y=\cdot4.$ 23. $x=-2\cdot5,\ y=-3\cdot5.$ 24. $x=\frac{1}{3},\ y=\frac{2}{3}.$
25. $x=\cdot02,\ y=2\cdot9.$ 26. $x=1\cdot5,\ y=2\cdot4.$ 27. 6.
28. 2, 6. 29. 1. 30. 5, 1.
31. $x=\frac{1}{4},\ y=1.$ 32. $x=\frac{1}{8},\ y=\frac{1}{2}.$ 33. $x=\frac{1}{2},\ y=1.$
34. $x=\frac{1}{5},\ y=\frac{1}{6}.$ 35. $x=\frac{1}{5},\ y=-\frac{1}{2}.$ 36. $x=1,\ y=1\frac{1}{4}.$
37. $x=3,\ y=4.$ 38. $x=\frac{1}{5},\ y=-\frac{1}{4}.$ 39. $x=\frac{1}{2},\ y=-\frac{1}{3}.$

X. c. (p. 87).

1. $x=2,\ y=3,\ z=-1.$ 2. $x=2,\ y=4,\ z=6.$ 3. $x=2,\ y=-3,\ z=4.$ 4. $x=-2,\ y=6,\ z=8.$
5. $x=-2,\ y=-3,\ z=-1.$ 6. $x=12,\ y=-24,\ z=12.$ 7. $x=3,\ y=-11,\ z=-10.$ 8. $x=9,\ y=3,\ z=6.$
9. $x=6,\ y=-2,\ z=-5.$ 10. $x=8,\ y=4,\ z=-3.$ 11. $x=-4\frac{1}{3},\ y=18,\ z=6\frac{1}{3}.$ 12. $x=12,\ y=24,\ z=36.$
13. $x=8,\ y=6,\ z=4.$ 14. $x=4,\ y=6,\ z=8.$ 15. $x=\frac{1}{2},\ y=\frac{1}{3},\ z=\frac{1}{4}.$ 16. $x=\frac{1}{3},\ y=\frac{1}{4},\ z=$ [illegible]
17. $x=5,\ y=11,\ z=17.$ 18. $x=40,\ y=45,\ z=48.$

XI. a. (p. 88).

21. $4x-2.$ 22. $4-x.$ 23. $2-2x.$ 24. $4x+$ [illegible]
25. $18-2x.$ 26. $a+b+c-d.$ 27. $4a-8.$ 28. $5y+$ [illegible]
29. $10a+7b.$ 30. $8c.$ 31. $a-2b.$ 32. [illegible]
33. $-24x+45.$ 34. $21x-42.$ 35. $3a+15.$ 36. [illegible]
37. $8a-16.$ 38. 10. [illegible] 39. 3[illegible] $-$ 5[illegible] 40. $30x$ [illegible]

XI. b. (p. 90).

1. $2a$. 2. c. 3. b. 4. $7x$. 5. $15-6x$.

6 $12-11a$. 7. $2b^2-2ab$. 8. $\frac{5}{8}-\frac{x}{8}$. 9. $2a-4b+24c+72d$

10. $-2x-2$. 11. x. 12. y. 13. 0. 14. $2x+y$.

23. $8a-3b$. 24. c. 25. $2a-6b$. 26. $3a$. 27. $2a$.

28. $2a-3b-6c$. 29. $-a+6b+72c+24d$. 30. $3a-7$.

31. $6xy+4y^2$. 32. $12a-2ab+4a^2b$. 33. x^2+3x.

34. $a+10b$. 35. $33a+28b$. 36. $26a-84$.

37. $18x-9xy \quad 9x^2y$. 38. $x-2x^3$.

XI. c. (p. 92).

1. $x^3+x^2(a+2)-x(6+2a)+a-7$. 2. $3x^2-2x(a+b+c)+a^2+b^2+c^2$.

3. $x^3+x^2(y+z)-x(y^2+z^2)-y^3$.

4. $-2x^3+3x^2(a+b)-3x(a^2+b^2)+a^3+b^3$.

5. $bx^3+x^2(a-b)-x(a+b)+a+c$. 6. $x^2(p^2-q^2)+2x(p-q)+p^2-q^2$.

7. $x^3(a-b)+x^2(c-b)+x(c-a)+d-e$.

8. $x^4(2-a)+x^3(6-a)+x^2(b-3)+x(-a-7)$.

9. $x^3+3x^2(y-z)+3x(z^2-y^2)+y^3$. 10. $x^3(a-c)+x^2(a-b)+x(c-b)+c$.

11. $x^4(a-p)+x^3(q-b)+x^2(r-c)$. 12. $x^2y(m+5n)+2xy^2(n-m)$.

13. $-x^3(b \quad a)-x^2(c-p)-x(d+q)-(p-c)$.

14. $-x^3(a+b)-x^2(b-a)-x(b-c)-(c-d)$.

15. $-x^3(b-a)-x(3a-4)+2a$.

XI. d. (p. 94).

17. 3. 18. 7. 19. 1. 20. 9.

21. $\frac{7x+5}{6}$. 22. $\frac{x}{12}$. 23. $\frac{7x-15}{10}$. 24. $\frac{2x+5}{35}$.

25. $\frac{7x+15}{20}$. 26. $\frac{7x-25}{12}$. 27. $\frac{5x}{12}$. 28. $\frac{5}{24}$.

29. $\frac{x+49}{30}$. 30. $\frac{9x+8}{12}$. 31. $\frac{9x+20}{36}$. 32. $\frac{9x}{40}$.

XII. a. (p. 95).

2. $15x^2-4xy-35y^2$, $-3y^2$. 3. 4.

4. $x=2$, $y=\ \ 2$. 5. $240a+30b+24c$, $\frac{a}{2}+\frac{b}{8}+\frac{c}{20}$.

6. 4 inches. 7. 48.

XII. b. (p. 95).

1. $\frac{91x-30}{60}$. 2. $3a+2b$, $9a^2-4b^2$.

3 -1. 4. $x=3$, $y=4$.

5. $\frac{a}{b}$ miles, $\frac{60b}{a}$ minutes, $\frac{bx}{a}$ hours. 6. 3·35 miles. 7. 96.

XII. c. (p. 96).

1. $11x+5$. 2. 3. 3. 5. 4. $x=13$, $y=2$.
5. $x+12$, $x-16$, 16, $40-x$ years. 6. 3·4 miles. 7. 51.

XII. d. (p. 96).

1. $46x-1$. 3. -7. 4. $x=2, 4, 6$. $y=1, 2, 3$.
5. $\frac{b}{a}$ pence, $\frac{bx}{a}$ pence, $\frac{12a}{b}$ lbs. 6. 36 feet. 7. 60, 47.

XII. e. (p. 96).

1. $ap+q$. 3. 1. 4. $x=2$, $y=-3$.
5. $\frac{x}{3}+14$, $13+x$, $2x$, $\frac{x}{4}$. 6. Half-a-mile, 9·04 miles. 7. 42, 32.

XII. f. (p. 97).

1. $x-2$, 2. 3. $-3\frac{5}{8}$. 4. $a=5$, $y=10$.
5. $5a$ pence, $\frac{3a}{5}$ pence, $\frac{240}{a}$ eggs. 6. 11·65 miles. 7. 50.

XII. g. (p. 97).

1. -44, -21, -6, 1, 0, -9, -26. 2. 225 lbs., 300 lbs.
3. $-\frac{1}{2}$. 4. 107, 117. 5. $x=5$, $y=-3$.
6. $x=-1$, $y=2$, $z=1$. 7. 62·5 feet nearly.

XII. h. (p. 98).

1. 46, 27, 14, 7, 6, 11, 22. 2. 570 sq. ft.
3. $1\frac{1}{2}$. 4. 7, -2. 5. 51, 53, 55, 57, 59.
6. 2·4 miles. 7. $x=-4$, $y=0$, $z=4$.

XIII. a. (p. 104).

1. $P_1(5, 4)$, $P_2(11, 8)$, $P_3(-5, 5)$, $P_4(-8, 9)$, $P_5(-9, -5)$, $P_6(-5, -3)$, $P_7(3, -5)$, $P_8(8, -7)$.
3. (i) (0, 0), (ii) (3, 0), (iii) (2, 2), (iv) $(-4, 4)$.
4. They all lie on a straight line parallel to OY.
5. 12. 6. 48. 7. 0, 12, 1·5, 3·5. 8. 18.
9. 36. 10. 74·6. 11. 180. 12. 3·5 sq. in.
13. 15. 14. 25. 15. 22. 16. 25 nearly.
17. 34. 18. 2·73 in. 19. 3·71 in. 20. 4·11 in.
21. 3·49 in. 22. 30. 23. 72. 24. 99.
25. 70. 26. 128. 27. 102. 28. 52.
29. 96. 30. 150. 31. 68 32. 95.

XIII. b. (p. 113).

30. $x=8$, $y=2$.
31. $x=3$, $y=2$.
32. $x=8$, $y=6$.
33. $x=4$, $y=0$.
34. $x=5$, $y=8$.
35. $x=4$, $y=3$.
36. $x=2\cdot8$, $y=4\cdot2$.
37. $x=4$, $y=5$.
38. $x=12$, $y=4$.
39. $x=7$, $y=17$
40. $x=9$, $y=12$.
41. $x=5$, $y=2$.
42. $x=10$, $y=5$.
43. $y=3x$.
44. $x-y=4$.
45. $2x+y=7$.
46. $y+5x=0$.
47. $y+5=2x$.
48. $y=3x+4$.
49. $2y=3x+12$.
50. $3y-x=5$.

XIV. a. (p. 117).

1. 17, 12.
2. 12, 5.
3. 6, 8.
4. 13, 9.
5. 4 pence, 9 pence.
6. 7 half-crowns, 3 florins.
7. 44 for, 31 against.
8. 24, 12.
9. 3*s*. 6*d*., 1*s*.
10. 20, 64.
11. 14, 38.
12. £450, £200.
13. 45, 15.
14. 7.
15. 14 florins, 11 half-crowns.
16. 63.
17. 72 miles, 5 miles an hour.
18. $2\frac{1}{2}$, $7\frac{1}{2}$ miles an hour.
19. 32*s*., 28*s*.
20. 57, 19.
21. 165.
22. 56, 67.
23. 17 florins, 7 half-crowns.
24. 93.
25. 9, 11 miles an hour.
26. 10, 30 gallons.
27. 100.
28. 15 miles, 2 miles an hour.
29. 24, 12, 4.
30. £51.
31. 24 bales, or 72 casks.
32. 12.
33. 24 feet long, 18 feet wide.
34. 5 teachers and 99 children at first, 7 teachers and 132 children at last
35. £13. 15*s*.
36. 81, 49 sq. yds.
38. 21 crowns, 40 half guineas.
39. 3.
40. 3 miles an hour.
41. 15 miles an hour, 90 miles.
42. 3 miles an hour, $8\frac{1}{2}$ miles.
43. 4 miles an hour, 24 miles.
44. 3000 ft. from the starting point.
45. £100, 5 pence in the £.
46. 3, 4, 5 miles an hour.
47. 300 miles; 150, 100 miles a day.

XIV. b. (p. 128).

1. 44 francs, 28 shillings.
2. 3 shillings, 20.
3. 38 minutes, 5 miles.
4. 13 ft. per sec., 17·5 ft. per sec., 2·5 secs.
5. 55 lbs., 84 lbs., 14·8 kilogrammes, 17·3 kilogrammes.
6. 4·9 c. in., 2·45 c. in., 41 c. cms.
7. 167°, 5°.
8. They meet at 3.30 P.M. 14 miles from Cambridge; 10 miles apart at 2.18 P.M., and 4.12 P.M.
9. In 10 secs. from *A*'s start, 33·3 yds. from the starting point
10. June, 1887.
11. 58, 38, 29.
12. 2·2 in., 12·45 cms.
13. 9·23 cms., 3·35 in.
14. 87, 78, [illegible]7, 51, 46, 42, 39, 38, 36, 17.
15. 2*s*. $2\frac{1}{2}d.$, 31 articles.
16. £1. 15*s*. 1*d*. approx.; 615 copies to the nearest 5.
17. £53.
18. 2·60, 5·63, 4·[illegible]6, 5·77.
19. £350; 4250 copies.

20. In half an hour from A's start, A having travelled 2 miles.

21. In $4\frac{1}{2}$ hours. **22.** 2·7 miles per hour.

23. ·25 of a mile per hour. **24.** $5\frac{5}{6}$ miles per hour.

25. In $2\frac{1}{2}$ hours, 20 miles from A's starting point; 2 hours, 3 hours.

26. In 3·1 hours, 24·8 miles from A's starting point; 2·6 hours, 3·6 hours from A's start.

27. $13\frac{1}{3}$ miles an hour. **28.** 35 miles, 45 miles.

XV. a. (p. 131).

1. $x^2+2ax-2bx+a^2-2ab+b^2$. **2.** $x^2-2ax-2bx+a^2+2ab+b^2$.

3. $a^2+2ab+b^2+4a+4b+4$. **4.** $a^2+b^2+c^2+2ab+2bc+2ca$.

5. $a^2+b^2+c^2-2ab+2bc-2ca$. **6.** $a^2+b^2+c^2-2ab+2ac-2bc$.

7. $a^2-2ab+b^2-4a+4b+4$. **8.** $4x^2+y^2+z^2+4xy+2yz+4xz$.

9. $x^2+4y^2+z^2-4xy+4yz-2zx$. **10.** $a^2+4b^2+9c^2+4ab+12bc+6ca$.

11. $a^2+4b^2+9c^2-4ab-12bc+6ca$. **12.** $9x^2+6ax-6bx+a^2-2ab+b^2$.

13. $4x^2+12ax-4bx+9a^2-6ab+b^2$. **14.** $4x^4+4x^3+5x^2+2x+1$.

15. $9x^4-6x^3+7x^2-2x+1$. **16.** $x^4+2x^3-15x^2-16x+64$.

17. $x^4+4x^3+6x^2+4x+1$. **18.** $x^4-2x^3-7x^2+8x+16$.

19. $4x^4-4x^3-19x^2+10x+25$. **20.** $x^2+2xy+y^2-6x-6y+9$.

21. $4x^2-4xy+y^2+16x-8y+16$. **22.** $1-2x+3x^2-2x^3+x^4$.

23. $4+4x-3x^2-2x^3+x^4$. **24.** $9-6x+13x^2-4x^3+4x^4$.

25. $25-20x+34x^2-12x^3+9x^4$.

26. $a^2+b^2+c^2+d^2+2ab+2ac+2ad+2bc+2bd+2cd$.

27. $a^2+b^2+c^2+d^2+2ab+2ac-2ad+2bc-2bd-2cd$.

28. $a^2+b^2+c^2+d^2-2ab+2ac-2ad-2bc+2bd-2cd$.

29. $a^2+b^2+4c^2+d^2+2ab+4ac+2ad+4bc+2bd+4cd$.

30. $a^2+b^2+4c^2+4d^2+2ab+4ac-4ad+4bc-4bd-8cd$.

31. $x^2+y^2+z^2+9+2xy+2yz+2zx-6x-6y-6z$.

32. $x^2+y^2+z^2+9-2xy+2yz-2zx+6x-6y-6z$.

33. $4x^2+y^2+4z^2+1-4xy-4yz+8xz-4x+2y-4z$.

34. $9a^2+4b^2+4c^2+d^2-12ab+12ac-6ad-8bc+4bd-4cd$.

35. $x^6+2x^5+3x^4+4x^3+3x^2+2x+1$.

36. $x^6+4x^5-6x^3+8x^2-4x+1$.

37. $x^6-2x^5+3x^4-4x^3+3x^2-2x+1$.

38. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.

XV. b. (p. 131).

1. $a^2-2ab+b^2-c^2$. **2.** $a^2+2ab+b^2-4c^2$.

3. $x^2+2xy+y^2-1$. **4.** $x^2+4xy+4y^2-b^2$.

5. $a^2-b^2-2bx-x^2$. **6.** a^2-4b^2+4bc- [illegible]

7. $4x^2+4ax+a^2-b^2$. **8.** $9y^2-a^2-2ab-b^2$.

9. $a^2-16x^2+8xy-y^2$. **10.** $1-a^2-2ab-b^2$.

11. $16-a^2+2ab-b^2$. **12.** $a^4+a^2b^2+b^4$.

13. $1-2a+a^2-b^2$.
14. $x^2+4xy+4y^2-b^2$.
15. $p^2-4q^2+12qr-9r^2$.
16. $1-4x^2+12xy-9y^2$.
17. $x^2+6xy+9y^2-16$.
18. x^4+x^2+1.
19. $1-4x+4x^2-49y^2$.
20. $4x^2+12xy+9y^2-25$.
21. $9x^4-x^2+4x-4$.
22. $4x^2-16y^2-40y-25$.
23. $25a^2+30a+9-4b^2$.
24. $a^4-2a^2b^2+b^4$.
25. $1+2x^2+9x^4$.
26. $a^2-2ab+b^2-c^2+2cd-d^2$.
27. $4x^2+4xy+y^2-a^2-2ab-b^2$.
28. $x^2+2ax+a^2-y^2+2by-b^2$.
29. $4x^2-4ax+a^2-y^2+4by-4b^2$.
30. $9x^2-12ax+4a^2-4y^2+12by-9b^2$.
31. $1-2x+x^2-y^2+2yz-z^2$.
32. $4-4a+a^2-9b^2+6bc-c^2$.

XV. c. (p. 134).

1. x^4-3x^2-6x+8.
2. $a^3+a^2b-ab^2-b^3$.
3. x^3-y^3.
4. $x^3+3x^2y-4xy^2-12y^3$.
5. $x^4-x^3-5x^2+27x-30$.
6. $x^4-6x^2-16x-15$.
7. $a^5-8a^4b+14a^3b^2+9a^2b^3-6ab^4$.
8. $-x^4-x^2y^2-y^4$.
9. $2a^4-7a^3b-4a^2b^2+23ab^3-6b^4$.
10. x^3-1.
11. x^3+8.
12. $8x^3-1$.
13. x^3-8y^3.
14. $27a^3+8b^3$.
15. x^3+1.
16. a^3+b^3.
17. x^3-8.
18. $x^3-4x^2y+3xy^2-12y^3$.
19. $x^4-5x^3+10x^2-7x-15$.
20. $x^4-13x^2-2x+35$.
21. $c^4-25c^2d^2-50cd^3-25d^4$.
22. $x^4+x^2y^2+y^4$.
23. $a^2b^2+c^2d^2-a^2c^2-b^2d^2$.
24. $-10a^4+21a^3b-21a^2b^2+16b^4$.
25. $x^4-2x^3-12x^2+x+2$.
26. $12x^4-34x^3+37x^2-17x+5$.
27. $20+11x-21x^2+7x^4-2x^5$.
28. $6+x-2x^2+7x^2y+7x^3y-3x^4y^2$.
29. $x^3+3x^2y+3xy^2+y^3-1$.
30. $x^6+3x^5-x^4-15x^3-14x^2+18x+24$.
31. $4x^5+3x^4-23x^3+25x^2-14x+4$.
32. $-5+8a-11a^2+4a^3+19a^4-9a^5-6a^6$.
33. $21x^4y-29x^3y^2+3x^2y^3+5xy^4$.
34. $6x^4-12x^2y^2+6y^4$.
35. $a^4+a^3b+ab^3+b^4$.
36. $a^3+b^3+c^3-3abc$.

XVI. a. (p. 136).

1. $x^2-5x+14$.
2. x^2-6x-5.
3. x^2-x+3.
4. $2x^2+2x+5$.
5. $3x^2-4x-5$.
6. $5+6x+4x^2$.
7. $x+1$.
8. $x-y$.
9. $x-2$.
10. $2x+1$.
11. $3a-2b$.
12. $5x-3y$.
13. $3x^2-2x+6$.
14. $x-1$.
15. x^2+xy+y^2.
16. $x-3$.
17. $9x^2+3x+1$.
18. a^2-ab+b^2.
19. x^3+x^2+x+1.
20. x^3-x^2+x-1.
21. x^2+1.
22. x^2+1.
23. $27x^3-18x^2+12x-8$.
24. x^2-x+1.
25. x^2+x+1.
26. x^2+2x+1.
27. x^2-4x+4.
28. $2x-4$.
29. a^2-a.
30. $12x^4-11x^3+10x^2+39x+8$.
31. $2x^3-3x^2+4x-5$.
32. $x^4-5x^3+13x^2-40x+119$.

XVI. b. (p. 138).

1. $a+2b+c$.
2. $a^2+2ab+2b^2$.
3. $a+b+c$.
4. $3a+2b+c$.
5. $x^4-ax^3+a^3x-a^4$.
6. $a-b+c$.
7. x^3+7x-5.
8. $x^2+xy-2x+y^2-4y+4$.
9. $a^8-a^7+a^5-a^4+a^3-a+1$.
10. $2x^2-3xy+5y^2$.
11. $a^2+b^2+c^2-ab-ac-bc$.
12. $a^2+b^2+c^2-ab+ac+bc$.
13. $x^2+y^2+4+xy+2y-2x$.
14. $x^5-x^4+x^3-x^2+x-1$.
15. x^2+xy+y^2.
16. $a+b+c$.
17. $ab+bc+ac$.
18. $x^6+x^4y^2+x^2y^4+y^6$.
19. $32a^5+16a^4+8a^3+4a^2+2a+1$.
20. $ab-ac-bc+c^2$.
21. $x^2+ax+3a^2$.
22. $x^2+2ax-4a^2$.
23. $\frac{x}{4}+\frac{3y}{2}$.
24. $\frac{a^2}{4}+\frac{ab}{6}+\frac{b^2}{9}$.
25. $\frac{x^2}{16}-\frac{xy}{20}+\frac{y^2}{25}$.
26. $\frac{a^2}{4}-\frac{ab}{6}+\frac{b^2}{9}$.
27. $\frac{a^2}{9}-\frac{2ab}{21}+\frac{b^2}{49}$.
28. $\frac{a}{5}-\frac{b}{4}$.

XVI. c. (p. 140).

1. -8.
2. 28.
3. -6.
4. -3427.
5. $-\frac{[illegible]}{4}$.
6. 35.
7. 11.
8. 10.
9. -9.
10. -53.
11. $-38\frac{1}{4}$.
12. 44.
13. $11\frac{3}{4}$.

XVII. a. (p. 141).

1. $x^2-2x(p+q+r)+(pq-qr+pr)$.
2. $127\frac{1}{2}$.
3. $x=5,\ y=6$.
4. 9 half-crowns, 3 threepenny pieces.
5. x^4+3x^2+4.
6. x^2+3y^2.
7. 3.

XVII. b. (p. 141).

1. $2x-y,\ 2x-y+20,\ 2x-y-20,\ y$.
2. 153.
3. Common roots, $x=6,\ y=8$.
4. 37.
5. $a^4+4a^3b+4a^2b^2-b^4$.
6. $16a^4-b^4$.
7. $2a^2-3ax+x^2$.

XVII. c. (p. 142).

1. $10x$ apples, $\frac{300}{x}$ pence.
2. 4.
3.

$x=-5$	-1	3	7	11	15
$y=7$	4	1	-2	-5	-8

4. $x^8+4x^6+6x^4+4x^2+1$.
5. $60x+18y+9z=480a$.
6. x^8-81y^8.
7. $3x^2-2x+3$.

XVII. d. (p. 142).

1. $\frac{60x}{y}$ yards, $\frac{1760y}{x}$ min.
2. 180.
3. $x=4{\cdot}07,\ y={\cdot}56$.
4. $x^6-3x^5-x^4+9x^3-5x^2-3x+2$.
5. $72,\ 74$.
6. $4ax^3+4abc$.
7. $a-3b-2$.

XVII. e. (p. 142).

1. xy miles, $60xy$ miles, $\frac{xy}{60}$ miles. 2. 226.
3. 162. 4. 12·57, 34·57, 62·86, 15, 10 inches.
5. $x^6-3x^5+6x^4-7x^3+6x^2-3x+1$. 7. $3a-b+4$.

XVII. f. (p. 143).

1. xy pence, $\frac{x}{3}$ pence, $\frac{x}{y}$ pence, $\frac{3x}{y}$ pence. 2. $\frac{1}{2}$.
3. $4y-11x=3$. 4. 26·7 miles. 5. $2x^4-11x^3+20x^2-14x+3$.
6. $4x^2+ab-ac-bc$. 7. $a-3b+4c$.

XVII. g. (p. 143).

1. $a+b$. 2. $x^2+2xy+y^2-z^2$. 3. 7.
4. 27 m. from one end, 18 m. from the other. 5. $x=5$, $y=11$.
6. $3x^2-2x+1$. 7. 56, 48.

XVII. h. (p. 144).

1. $x-x^2$. 2. $a^2x^2-2a^2x+a^2$. 5. 11, 7.
6. $3x-7y$. 7. 21.

XVII. k. (p. 144).

1. x^2+7. 2. -39, -20, -7, 0, 1, -4, -15. 3. $-2\frac{1}{2}$.
4. 22 miles, 48 minutes. 5. $x=2\frac{2}{5}$, $y=12$.
6. $2x^2+3x+1$. 7. $x=2$, £5. 5*s*.

XVII. l. (p. 144).

1. x^8-y^8. 2. $x+y+z-3a$. 3. $-6\frac{1}{3}$.
4. 1·69 in., 2·25 in., 3·8 cms., 5·58 cms. 5. $2x^2-5x-3$.
6. 180. 7. $x=3$, $y=1$, $z=5$, $w=9$.

XVIII. a. (p. 145).

1. $a(x+b)$. 2. $a(x-a)$. 3. $x(x-3a)$.
4. $x^2(x-5a)$. 5. $a(x^2-ax+a^2)$. 6. $3a(a-b)$.
7. $5x^2(x-3y)$. 8. $x(x-y)$. 9. $7(3-8x)$.
10. $5x(5x-4y)$. 11. $x(a-b+c)$. 12. $-2x(x^2-2)$.
13. $-y(a-b-c)$. 14. $px(px-ay+by)$. 15. $19a^2x^2(4x-3a)$.
16. $3(p^2x^2-3px+4)$. 17. $xyz(x+y-z)$. 18. $7b(a-c-3x)$.
19. $7x(2x^2-xy+8y^2)$. 20. $6xyz(6x-9y+8z-3xyz)$.

XVIII. b. (p. 147).

1. $(x+4)(x+5)$. 2. $(x-3)(x-7)$. 3. $(x+4)(x+6)$.
4. $(x+3)(x+7)$. 5. $(x-4)(x-6)$. 6. $(x-1)(x-7)$.

7. $(x+1)(x+2)$. 8. $(x-2)^2$. 9. $(x-2)(x+1)$.
10. $(x+2)(x-1)$. 11. $(x+1)^2$. 12. $(x+5)(x-1)$.
13. $(x-5)(x+1)$. 14. $(x+5)(x+7)$. 15. $(x-3)^2$.
16. $(x-10)(x-1)$. 17. $(x-3)(x-9)$. 18. $(x+3)(x+17)$.
19. $(x-5)(x-13)$. 20. $(x-5)^2$. 21. $(x+7)(x-6)$.
22. $(x-7)(x+6)$. 23. $(x+9)(x-5)$. 24. $(x-7)(x+5)$.
25. $(x+7)^2$. 26. $(x+9)(x-7)$. 27. $(x-12)(x-10)$.
28. $(x-13)(x+10)$. 29. $(x+9)(x-8)$. 30. $(1-2x)(1-x)$.
31. $(7+x)(3+x)$. 32. $(x+p)(x+q)$. 33. $(x-m)(x-n)$.
34. $(x+m)(x-n)$. 35. $(x-m)(x+n)$. 36. $(x+2a)(x+b)$.
37. $(x-a)(x-3b)$. 38. $(x-2a)(x+3b)$. 39. $(x+4a)(x-5b)$.
40. $(x-5a)(x+3b)$. 41. $(x-2)(x+9)$. 42. $(x-11)(x+10)$.
43. $(1-3x)(1-2x)$. 44. $(5+x)(1-x)$. 45. $(x+17)(x-1)$.
46. $(8-x)(5-x)$. 47. $(1+10x)(1-13x)$. 48. $(x-15)(x+1)$.
49. $(8+x)(5-x)$. 50. $(x+11)(x-10)$. 51. $(7+x)(6-x)$.
52. $(6+x)(11-x)$. 53. $(1-6x)(1-x)$. 54. $(9-x)(8+x)$.
55. $(x-8)(x-27)$. 56. $(x+10y)(x-y)$. 57. $(a+15b)(a+b)$.
58. $(x-11)(x-12)$. 59. $(5x+y)(x-y)$. 60. $(a-6b)(a+4b)$.
61. $(x-11y)^2$. 62. $(x-15)^2$. 63. $(x-72)(x-1)$.
64. $(x-13y)^2$. 65. $(x-102)(x-1)$. 66. $(73x-1)(x-1)$.
67. $(x-9a)(x-5a)$. 68. $(9x+y)(6x-y)$. 69. $(13x-1)(2x+1)$.
70. $(16x-1)(15x+1)$. 71. $(43x+1)(x-1)$. 72. $(1-3ab)(1-2ab)$.
73. $(xy-8)(xy+4)$. 74. $(13x+1)(12x-1)$. 75. $(1-5xy)^2$.
76. $(17xy-1)(3xy-1)$. 77. $(7ab+1)(6ab-1)$. 78. $(17x-y)(x+y)$.
79. $(18x+y)(3x+y)$. 80. $(18x+y)(3x-y)$. 81. $(19-x)(3-x)$.
82. $(xy-5)(xy-11)$. 83. $(xy-16)(xy+3)$. 84. $(x-92)(x-1)$.
85. $(167+x)(1-x)$. 86. $(x-17)^2$. 87. $(1-15x)^2$.
88. $(81x+1)(x+1)$. 89. $(x-13y)(x+3y)$.

XVIII. c. (p. 148).

1. $(a+b)(x+y)$. 2. $(a-b)(x-y)$. 3. $(x-y)(a-2)$.
4. $(x-y)(6-a)$. 5. $(x+z)(x+y)$. 6. $(x-y)(x+z)$.
7. $(ac+b)(ac-d)$. 8. $(x+y)(x-2)$. 9. $(x-y)(3-a)$.
10. $(a-c)(a-b)$. 11. $(b+a)(c-a)$. 12. $(ac+d)(ac+b)$.
13. $(a^2+b^2)(c+d)$. 14. $(a^2+b^2)(c-d)$. 15. $(x-3)(x^2+2)$.
16. $(x-2)(x^2-y)$. 17. $(x+5)(x^4-3)$. 18. $(x^2+1)(y^2+1)$.
19. $(x-1)(y^2+1)$. 20. $(ax-b)(bx-a)$. 21. $(x-y)(x+y-4)$
22. $(a+m)(a+m^2)$. 23. $(x+1)(x^2+1)$. 24. $(x+1)(x^4+1)$.
25. $(2x-1)(x^2+1)$. 26. $(a-b)(x^2+1)$. 27. $(2x-3)(x^2+2)$.
28. $(3x-1)(x^2+4)$. 29. $(7x-3)(x^2-3)$. 30. $(2x-1)(x^2-5)$.
31. $(x+7)(2x^2-3)$. 32. $(x+5)(11x^2+7)$. 33. $(a^2-b)(c+1)$.
34. $(x+1)(x-a^2)$. 35. $(2+x)(a-x^2)$. 36. $(x+3)(2x^2-c)$.

XVIII. d. (p. 149).

1. $(1-x)(1+x)$.
2. $(1-2x)(1+2x)$.
3. $(x-2a)(x+2a)$.
4. $(a-7)(a+7)$.
5. $(3a+x)(3a-x)$.
6. $(3x+1)(3x-1)$.
7. $(5x-4)(5x+4)$.
8. $(x+3)(x-3)$.
9. $(5x-7)(5x+7)$.
10. $(a-5)(a+5)$.
11. $(11-b)(11+b)$.
12. $(a-3)(a+3)$.
13. $(x-13)(x+13)$.
14. $(2-a)(2+a)$.
15. $(4-11x)(4+11x)$.
16. $(ab+cd)(ab-cd)$.
17. $(3xy+4ab)(3xy-4ab)$.
18. 100×102.
19. 8×14.
20. $(xy+1)(xy-1)$.
21. $(8-cd)(8+cd)$.
22. $(1-3k)(1+3k)$.
23. $(3-2a)(3+2a)$.
24. $(3ab-4)(3ab+4)$.
25. 1×305.
26. $(x-100)(x+100)$.
27. $(100x+1)(100x-1)$.
28. $(xy-9a^2)(xy+9a^2)$.
29. $(a^3-b^2)(a^3+b^2)$.
30. $(b^2+5)(b^2-5)$.
31. $(x^4+a)(x^4-a)$.
32. $(6x^6-y^4)(6x^6+y^4)$.
33. $(ab^3c^2-x)(ab^3c^2+x)$.
34. $(1-10x)(1+10x)$.
35. $(abc+d)(abc-d)$.
36. $(1-11a^2)(1+11a^2)$.
37. $(7x-6y)(7x+6y)$.
38. $(pq-2)(pq+2)$.
39. $(12x^2+y^2z^3)(12x^2-y^2z^3)$.
40. $(a-15b)(a+15b)$.
41. $(9x-8)(9x+8)$.
42. $(2mn+1)(2mn-1)$.
43. $(3p-7q)(3p+7q)$.
44. $(x-13y)(x+13y)$.
45. $(9ab+1)(9ab-1)$.
46. $(x^{18}-y^9)(x^{18}+y^9)$.
47. $(a-17b)(a+17b)$.
48. $(11a+12b)(11a-12b)$.
49. $(5x^8-13a^5)(5x^8+13a^5)$.
50. $(x^2y-10)(x^2y+10)$.
51. $(xy^2-12p)(xy^2+12p)$.
52. $(1-10x^3y^2z^4)(1+10x^3y^2z^4)$.
53. $(11x^3y^4-1)(11x^3y^4+1)$.
54. 67,000.
55. 1800.
56. 998,000.
57. 640.
58. 1002,000.
59. 54,800.
60. 33,096.
61. 136.
62. 650,000.
63. 573.
64. 313,800.
65. 996,000.
66. 15,152.
67. 9,400.
68. 43,984.
69. 11,800.
70. 9,999,800,000.
71. 13,440.
72. 15,000.
73. 15,600.
74. 59,600.
75. 128,400.

XVIII. e. (p. 150).

1. $3(x-2a)(x+2a)$.
2. $7(1-x)(1+x)$.
3. $2(x-12)(x+12)$.
4. $5x^2(3y-4a)(3y+4a)$.
5. $3(a^4+x)(a^4-x)$.
6. $7a^2y(4xy-5)(4xy+5)$.
7. $6(3ab+2cd)(3ab-2cd)$.
8. $141a^3b^3(a^3b^2-2)(a^3b^2+2)$.
9. $7(a-7b)(a+7b)$.
10. $3(5x-4)(5x+4)$.

11. $11(1-3b)(1+3b)$. **12.** $5(3ab-4)(3ab+4)$.
13. $13(a^3-b)(a^3+b)$. **14.** $7(x-15a)(x+15a)$.
15. $3(x^2-10)(x^2+10)$. **16.** $3a(3p-7q)(3p+7q)$.
17. $5c(11x+12b)(11x-12b)$. **18.** $13ab(c-2d)(c+2d)$.
19. $17(1-2pq)(1+2pq)$. **20.** $7x^2y^2(1-2y)(1+2y)$.

XVIII. f. (p. 151).

1. $(a-b+c)(a-b-c)$. **2.** $(a+b+c)(a-b-c)$.
3. $(x-y+2a)(x-y-2a)$. **4.** $(x+2y+4b)(x+2y-4b)$.
5. $(x+2a-b)(x-2a+b)$. **6.** $(x+y+a+b)(x+y-a-b)$.
7. $(3x+4y)(x+2y)$. **8.** $(a+4x-y)(a-4x+y)$.
9. $(5x+a-b)(5x-a+b)$. **10.** $(4a+5x+5y)(4a-5x-5y)$.
11. $4x$. **12.** $8ax$. **13.** $(a-2b+c+d)(a-2b-c-d)$.
14. $(a+b+c+x+y+z)(a+b+c-x-y-z)$.
15. $(4x+y)(2x-3y)$. **16.** $16(2x+1)$.
17. $20pq$. **18.** $y(6x-y)$.
19. $(2x+2a+3y+3b)(2x+2a-3y-3b)$. **20.** $(5x+y)(x+5y)$.
21. $3(a+b+2c+2d)(a+b-2c-2d)$. **22.** $(8p+q-4)(8p-q+4)$.
23. $4ab$. **24.** $(3x+2y+2a)(x+4y)$.
25. $5(x+y)(x-y)$. **26.** $-48ax$.
27. $(1+3x-2y)(1-3x+2y)$. **28.** $(1+2x-2y)(1-2x+2y)$.
29. $(10+2a-3b)(10-2a+3b)$. **30.** $b(8a-b)$.
31. $(a-b)^2(a+b)^2$. **32.** $(a^2+2ab+2b^2)(a^2-2ab+2b^2)$.
33. $2ab-1$. **34.** $5(a-1)(a+1)$. **35** $(2x^2+1)(5-4x)$.

XVIII. g. (p. 151).

1. $(a-b+c)(a-b-c)$. **2.** $(c+a+b)(c-a-b)$.
3. $(x+a+b)(x+a-b)$. **4.** $(y+a-x)(y-a+x)$.
5. $(a+b-c)(a-b+c)$. **6.** $(1+a-b)(1-a+b)$.
7. $(x+a-y)(x+a+y)$. **8.** $(x-2y+3ab)(x-2y-3ab)$.
9. $(x-y+3)(x-y-3)$. **10.** $(4+a-b)(4-a+b)$.
11. $(1+2a-b)(1-2a+b)$. **12.** $(a+x+b+y)(a+x-b-y)$.
13. $(2a-b+x+c)(2a-b-x-c)$. **14.** $(a-b+c-d)(a-b-c+d)$.
15. $(a-c+b+d)(a-c-b-d)$. **16.** $(x^2+x+1)(x^2-x-1)$.
17. $(a+c+b)(a+c-b)$. **18.** $(3a-b+x+2c)(3a-b-x-2c)$.
19. $5(a-b+2c)(a-b-2c)$.

XVIII. h. (p. 154).

1. $(5x-2)(x-2)$. **2.** $(x+3)(3x+5)$. **3.** $(x-2)(3x-1)$.
4. $(x+7)(2x-3)$. **5.** $(x-6)(3x+5)$. **6.** $(x+9)(5x-3)$.
7. $(x+9)(2x+1)$. **8.** $(x-7)(3x-1)$. **9.** $(2x-5)(2x-3)$.
10. $(3x-2)(3x-4)$. **11.** $(4x+3)(4x-5)$. **12.** $(7x+1)(7x+2)$.

13. $(3x-2)(3x+4)$. **14.** $(2x-7)(2x+9)$. **15.** $(2x+3)(3x+1)$.
16. $(2x-3)(3x-1)$. **17.** $(3x-2)(2x+1)$. **18.** $(4x-3)(3x-4)$.
19. $(5x+4)(4x+5)$. **20.** $(3x-4)(4x+3)$. **21.** $(6x+1)(3x-2)$.
22. $(4x-5)(6x-5)$. **23.** $(1-2x)(3-2x)$. **24.** $(5-x)(1+2x)$.
25. $(2x+3y)(x+y)$. **26.** $(2x-y)(x+2y)$. **27.** $(6x-5y)(2x+3y)$.
28. $(7x-3)(2x+5)$. **29.** $(3x-7)(3x+4)$. **30.** $(7x-4)(2x-3)$.
31. $(5x-9y)(2x+y)$. **32.** $(7x-3y)(x+y)$. **33.** $(12x+5y)(x+y)$.
34. $(13x-1)(2x-3)$. **35.** $(13x+2)(x+3)$.

XVIII. k. (p. 155).

1. $(x+y)(x^2-xy+y^2)$.
2. $(x-y)(x^2+xy+y^2)$.
3. $(1-x)(1+x+x^2)$.
4. $(1+x)(1-x+x^2)$.
5. $(x^2+y)(x^4-x^2y+y^2)$.
6. $(x^2-y)(x^4+x^2y+y^2)$.
7. $(2x-1)(4x^2+2x+1)$.
8. $(1+2y)(1-2y+4y^2)$.
9. $(2a+b)(4a^2-2ab+b^2)$.
10. $(1+3x)(1-3x+9x^2)$.
11. $(x+3)(x^2-3x+9)$.
12. $(y-3)(y^2+3y+9)$.
13. $(a+5)(a^2-5a+25)$.
14. $(5a-1)(25a^2+5a+1)$.
15. $(2x-3y)(4x^2+6xy+9y^2)$.
16. $(2a+3b)(4a^2-6ab+9b^2)$.
17. $(a-6)(a^2+6a+36)$.
18. $(7x-1)(49x^2+7x+1)$.
19. $(y-4)(y^2+4y+16)$.
20. $(4+y)(16-4y+y^2)$.
21. $(10x+1)(100x^2-10x+1)$.
22. $(ab-1)(a^2b^2+ab+1)$.
23. $(1+ab)(1-ab+a^2b^2)$.
24. $(ab^2-4)(a^2b^4+4ab^2+16)$.
25. $(2xy-1)(4x^2y^2+2xy+1)$.
26. $(x^2+1)(x^4-x^2+1)$
27. $(4a-5b)(16a^2+20ab+25b^2)$.
28. $(3r+pq)(9r^2-3pqr+p^2q^2)$.
29 $(6a-b)(36a^2+6ab+b^2)$.
30 $(8x+1)(64x^2-8x+1)$.
31 $(9a-2x)(81a^2+18ax+4x^2)$
32. $(1+9x)(1-9x+81x^2)$.
33 $(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
34. $(x-2)(x+2)(x^2+2x+4)(x^2-2x+4)$.

XVIII. l. (p. 155).

1. $-8x(x^2-2)$.
2. $(a-6b)(a-5b)$.
3 $3(x-1)(x+1)$.
4. $3a^3b^3c^2(a^2-7bc+6ab)$.
5 $3(a-3)(a+3)$.
6. $5(a-2)(a^2+2a+4)$.
7. $(10a-b)(a+b)$.
8. $3(2a-3)$.
9. $xy(x^4-3y^4)$.
10. $7(a-5)(a+5)$.
11. $-(1+x)(1+x^2)$.
12. $11ac(c-3a)$.
13. $3(1-6x)(1-x)$.
14. $3(a-1)(a+1)(b-1)(b+1)$.
15. $3(2+x)(2-x)$.
16. $p^4q^4r^4(p^2q^3-3qr^4+2p^5)$.
17. $3\times 14\times 8=3\times 7\times 2^4$.
18. $3(5x-2)(x-2)$.
19. $(x-p)(x+q)$.
20. $4(x-10y)(x+y)$.
21. $5(1-3y)(1+3y)$.
22. $10(2x-y)(x+2y)$.
23. $11(x-11y)(x-12y)$.
24 $3(1-3x)(1+3x+9x^2)$.
25. $(5-x)(x-1)$.
26. $(x-y)(x-y-5)$.

27. $15(x-y)(x+y)(x^2+y^2)$.
28. $3(x-1)^2$.
29. $3(a-2)(b-c)$.
30. $13(3x+1)(3x-1)$.
31. $2(x-5)(x^2+5x+25)$.
32. $(px+1)(qx+1)$.
33. $2(x-1)(x-7)$.
34. $7(x+y)(x-2)$.
35. $2(a-5)(a+5)$.
36. $(a+7b)(a-6b)$.
37. $2(3x+2y)(3x-2y)$.
38. $3p^2q^2(5q-4p+6)$.
39. $3(11+x)(11-x)$.
40. $9(x+5)(x-1)$.
41. $(4x-1)(6x+1)$.
42. $(1+x)(1-x)(2+x)$.
43. $5(x-y)(x^2+xy+y^2)$.
44. $3(x+5)(x+4)$.
45. $3(xy-1)(x^2y^2+xy+1)$.
46. $5(2pq+1)(2pq-1)$.
47. $a(2bc-1)(4b^2c^2+2bc+1)$.
48. $17(x+1)(x+2)$.
49. $(5a-3b-2c)(a-3b+2c)$.
50. $7(xy^2+10)(xy^2-10)$.
51. $2(x-y+1)(x-y-1)$.
52. $3(1+x-y)(1-x+y)$.
53. $(1-2x)(1-3x)$.
54. $(x-5y)(x-4y)$.
55. $3(a-b)(a+b)$.
56. $(1+2x-2y)(1-2x+2y)$
57 $13x(3x-2)$.
58. $2(x+5y)(x+7y)$.
59. $12x(1-x)$.
60. $(x-15)^2$.
61 $x(6x+1)(3x-2)$.
62. $3(x-2)(x+2)$.
63 $x(5-x)(1+2x)$
64. $15ab(a-2b)$.
65. $x^2(3x-2)(2x+1)$.
66. $(7x-1)(x-1)$.
67. $5(8+x)(5-x)$.
68. $2abc(2a-3b+4c)$.
69. $7(x+1)(x-1)$.
70. $x(x-3)(x^2+3x+9)$.
71. $(x+7y)(x-6y)$.
72. $9(x-7)(x+5)$.
73. $x(a-5)(a^2+5a+25)$.
74. $x(1-2x)(3-2x)$.
75 $(2a+b)^2$.
76. $7(a+11)(a-10)$.
77. $x^2(13x+2)(x+3)$
78. $(x+p)(x-q)$.

XVIII. m. (p 157).

1. $(a-b)(a+b)(a^2+b^2)$
2. $(2a-1)(2a+1)(4a^2+1)$
3. $2(2x-y)(2x+y)(4x^2+y^2)$.
4. $(x^2+x-1)(x^2-x+1)$.
5. $3a(x-a)(x+a)(x^2+ax+a^2)(x^2-ax+a^2)$.
6. $28ab$
7. $(a-b+2c-2d)(a-b-2c+2d)$.
8. $4ab(a-b)^2$.
9. $(x-y)(x-y+1)(x-y-1)$.
10. $(x-3)(2x-1)(2x+1)$.
11. $(x-3)(x+3)(2x+1)$.
12. $(ax-by)(bx-ay)$.
13. $(a+c)(b-d)$.
14. $(2x^2+3y^2)(2x-3y)(x+y)$.
15. $(x-2)(x+2)(x+3)(x-3)$.
16. $a^2b^2(1+ab)(1-ab+a^2b^2)$.
17. $a(a-b+c)(a-b-c)$.
18. $(x-a)^2(x+2a)$.
19. $(6x-1)(14x+1)$.
20. $(7x-3)(x+15)$.
21. $(1+x+x^2)(1+x-x^2)$.
22. $b(a+b-c)(a-b+c)$.
23. $(a^2+4b^2)(a-2b)(a+2b)$.
24. $(a-1)(a+1)(a^2+a+1)(a^2-a+1)$.
25. $(x-1)(x+1)(x-2)(x+2)$.
26. $(x+y)^3(x-y)$.
27. $(x+1)(x-a)$.
28. $(x+3a+b)(x-a)$.
29. $(x+3a+b)(x-b)$.
30. $[x(a+b)+y(a-b)][x(a-b)-y(a+b)]$.

31. $(x+1)(x^2+1)(x^4-x^2+1)$. **32.** $(20x+7)(10x-3)$.
33. $(x+y+z)(x-y-z)(x-y+z)(x+y-z)$.
34. $9(x-y)(x^2-xy+y^2)$. **35.** $x(x+1)(x-2)(x+5)$.
36. $(x+b)(bx+a^2)$. **37.** $(x+1)(2x-5)(x-3)$.
38. $(x^2+y^2)(a^2+b^2+c^2)$. **39.** $(3x-5)(5x+7)$.
40. $(x-b)(bx+a^2)$. **41.** $4ab(1+a)(1-a)(1+b)(1-b)$.
42. $[ax-(a-1)][(a+1)x+a]$. **43.** $(x-1)(x-2)^2(x+2)$.
44. $(x-1)^2(x+1)(5x+1)$. **45.** $(x+y)(3x-2y)(2x-5y)$.
46. $(x-3)(x^2-x+1)$. **47.** $[(a+2)x+a+1][ax-(a-1)]$.
48. $(a-b)(a+ab+b)$. **49.** $(2a+b-c)(2a-b+c)(4a^2+\overline{b-c}|^2)$.
50. $3(a+b+c)(b-c)$. **51.** $(x+y)(5x-3y)(3x-2y)$.
52. $(x-2)(x^2+2x-2)$. **53.** $(x-y)^3(x+y)$.
54. $(x+ay)(x-by)$. **55.** $(5p-4q)(p-3q)$.
56. $x(1+2ay)(1-2ay+4a^2y^2)$. **57.** $3(3x^2-4y)(3x^2+4y)$.
58. $(x-2)(x^2+x+2)$. **59.** $(2x-5)(x+6)$.
60. $(a-x)(1+ax)$. **61.** $xy(y+x)(y-x)(y^2+x^2)$.
62. $4(x-12)(x+9)$. **63.** $(b-1+a)(b-1-a)$.
64. $(x-1)(x+1)(x+3)(x-3)$. **65.** $(5x-1)(11-x)$.
66. $(x-3)(x+2)(x-2)(x+1)$.

XIX. a. (p. 159).

1. $5ab$. **2.** x^2y^2. **3.** ab. **4.** $2xyz$.
5. $3a^2bc^2$. **6.** $3x^3$. **7.** $3xy$. **8.** y.
9. $3a^2c^2$. **10.** $13x^2$. **11.** $5a^3d$. **12.** abc.

XIX. b. (p. 160).

1. a. **2.** $x-2$. **3.** $x+y$. **4.** $x-2$.
5. $a+2b$. **6.** $x+y$. **7.** $x-2y$. **8.** $x+y$.
9. $x(x-3a)$. **10.** $3(x-3)$. **11.** $x+4y$. **12.** $x-2y$.
13. $x+1$. **14.** $1-x$. **15.** $1+x$. **16.** $x-3$.
17. $x+y$. **18.** $x+4$. **19.** $x+11$. **20.** $x+5$.
21. $x+a$. **22.** a^2-ab+b^2. **23.** $x-6$. **24.** $x-3$.
25. $3a^3b^2(a+b)$. **26.** $3x-1$. **27.** $x+3$. **28.** $(x-1)(x-2)$.
29. $a+b+c$. **30.** $5x-1$. **31.** $x-2$. **32.** $(a-b+c)$.
33. $x-5$. **34.** $x-a$. **35.** $2x-1$. **36.** $4x^2-6x+9$.
37. $x-1$. **38.** $x-1$. **39.** $(x-1)(3x-2)$. **40.** $x-5$.

XIX. c. (p. 163).

1. $a(3x^2-2ax+a^2)$. **2.** x^2+xy+y^2. **3.** $2x^2-x-3$.
4. $x-2$. **5.** $x+2$. **6.** $x+4$.
7. x^2+5x+1. **8.** $4x+3$. **9.** $2x-5$.
10. x^2-5x+1. **11.** $2x+7$. **12.** $x+2$.
13. $x-4$. **14.** $2x^2+7x+3$. **15.** x^2-3.
16. $3x^2+y^2$. **17.** $x^3-3x^2y+3xy^2-y^3$.
18. $5x^2-1$. **19.** x^2+x+2. **20.** x^2+8x-2.

XIX. d. (p. 165).

1. $\frac{a^2}{2}$. 2. $\frac{2x^2}{a}$. 3. $\frac{5a}{12c}$. 4. $\frac{3x^2z^2}{4y^2}$. 5. $\frac{3b^2}{2a^2}$.

6 $\frac{5mp^4}{2n^2}$. 7. $\frac{a}{a+b}$. 8. $\frac{x}{x-y}$. 9. $\frac{3x}{4x-3y}$. 10. $\frac{1}{x-y}$.

11. $\frac{3a}{4b}$. 12. $\frac{2(2x-3y)}{3x-2y}$. 13. $\frac{3}{5}$. 14. $\frac{b}{c}$.

15. $\frac{xy}{3bz}$. 16. $\frac{x}{x-3}$. 17. $\frac{x}{2-x}$. 18. $\frac{x+2}{x+3}$.

19. $\frac{1+2x}{1-3x}$. 20. $\frac{x+b}{x+c}$. 21. $\frac{a+b}{a^2+ab+b^2}$. 22. $\frac{x-y}{x+y}$.

23. $\frac{b-a}{b+a}$. 24. $\frac{1+bx}{1+cx}$. 25. $\frac{2(x-3)}{3(x-2)}$. 26. $\frac{x^2-1}{x^2+1}$.

27. $\frac{x+b}{x-c}$. 28. $\frac{x^3-y^3}{x^3+y^3}$. 29. $\frac{x-5}{2x+3}$. 30. $\frac{a+b-c}{a-b-c}$.

31. $\frac{3x-1}{x^2-1}$. 32. $\frac{a+b-c-d}{a-b+c-d}$. 33. $\frac{x-5}{x-3}$. 34. x^2-x+1.

35. $\frac{x^2+5x+5}{x^2+x-2}$. 36. $\frac{x+1}{3x^2+3x+10}$. 37. $\frac{x^2+3a}{x^2-3a}$. 38. $\frac{x(x+5)}{x^2+x-5}$.

39. $\frac{x+y-1}{x+y+1}$. 40. $\frac{a-1}{a+1}$. 41. $\frac{3a-4b}{3a+2b}$. 42. $\frac{2-x-y}{2+x-y}$.

43. $\frac{3a-2b}{3a+2b}$. 44. $\frac{2a+b-c}{2a-b-c}$. 45. $\frac{9-3a+a^2}{3}$. 46. $\frac{3x+2}{3x-2}$.

XIX. e. (p. 167).

1. $\frac{y}{x}$. 2. $\frac{x-7}{x-3}$. 3. 1. 4. $\frac{2x-1}{2y+1}$. 5. $\frac{x+1}{x+2}$.

6. $\frac{x+b}{x+c}$. 7. $\frac{x+2}{x+5}$. 8. $a(x+a)$. 9. 1. 10. 1.

11. $\frac{x-5}{x-6}$. 12. x. 13. $\frac{x+5}{x+3}$. 14. $\frac{3x}{x+6}$.

15. $\frac{a-b+c}{a+b+c}$. 16. $\frac{4x(x-3)}{x+5}$. 17. $\frac{2x+3}{3x-1}$. 18. $\frac{a^2+ax+x^2}{a^2-ax+x^2}$.

19. $\frac{6}{x-3}$. 20. $\frac{a-b+c}{ab}$. 21. $\frac{x(1+6x)}{1-6x}$. 22. $\frac{2(x+7)}{x+5}$.

23. $\frac{x^2-x+1}{x(x+9)}$. 24. $\frac{x-1}{x+1}$. 25. $\frac{x^2-ax+a^2}{x^2+ax+a^2}$. 26. $\frac{7x-3y}{3(3x-7y)}$.

XX. a. (p. 168).

1. a^2b^2c. 2. $4a^2x^2$. 3. $12a^5$. 4. $30x^2y^2$.

5. $294x^2y^2z$. 6. $2a^2b^2$. 7. $60x^4y^3$. 8. xyz.

9. $24a^3b^4$. 10. $12a^4b^4$. 11. $108x^4y^3$. 12. $a^2y^2z^2$.

13. $60a_4$ 14. a^3b^3. 15. $36a^3b^4c^4$. 16. $240x^6y^4z^6$.

XX. b. (p. 169).

1. $4x(a-x)$.
2. $a^2(a-b)$.
3. $6(a-x)(a+x)$.
4. $21(a+b)$.
5. $a^2b^2(a-b)$.
6. $xyz(x-y)$.
7. $4x^2y(x+y)$.
8. $6(x-1)(x+1)$.
9. $a^2(a-x)$.
10. $4a^2x(a+x)$
11. $15(a-b)$.
12. $12(x-y)(x+y)$.
13. $6x^2(x^2+1)^2$.
14. $12(ax-by)(ax+by)$.
15. $xy(x+y)(x-y)$.
16. $8(1-x)(1+x)(1+x^2)=8(1-x^4)$.
17. $12(x-1)(x^2+x+1)=12(x^3-1)$.
18. $(x+1)(x+2)(x+3)$.
19. $(x-1)^2(x+2)$.
20. $(x-2)(x-3)(x-7)$
21. $(x+1)(x-4)(x+6)$.
22. $(a+b+c)(a+b-c)(a-b+c)$.
23. $18(x+y)^3$.
24. $(2x-1)(x-3)(x+3)$.
25. $(3x-1)(x-2)(x+3)$.
26. $(x^2-y^2)^2$.
27. $(x+6y)(x-6y)(x+y)(x-y)$
28. $105ab(a+b)(b-a)$.
29. $12x^2(x+y)(x-y)$.
30. $72x^2y^3(x-1)(x-2)^2$.
31. $a(a-b)(2a-b)(a^2+ab+b^2)$.
32. $(2x-1)(x-3)(3x+2)$.
33 $(x+1)(x-2)(x-3)$.
34. $(x-2)(x+2)(x-1)(x+1)$.
35 $18a^2b^2(a-b)(a+b)(a^2+ab+b^2)(a^2-ab+b^2)$.
36. $36x(x-y)(x+y)(x^2+xy+y^2)(x^2-xy+y^2)$.
37. $(x-2a)(x+2a)(x^2+4a^2)$.
38. $(x-a)(x-b)(x+3a+b)$.
39. $(x-3)(x+3)(2x-1)(2x+1)$.
40 $(a-b)(b-c)(c-a)$.

XXI. a. (p. 170).

1. $\frac{11}{6x}$.
2. $\frac{5a}{6x}$.
3. $\frac{bc+ca+ab}{abc}$.
4. $\frac{bc+ca-ab}{abx}$.
5. $\frac{a^2+b^2+c^2}{abc}$.
6. $\frac{x}{12}$.
7. $\frac{x-6}{42}$.
8. 1.
9. $\frac{bx-ax}{ab}$.
10. $\frac{-z+2x}{xz}$.
11. $\frac{2}{15}$.
12. $\frac{2p+3r}{6pr}$.
13. $\frac{13x+2}{12}$.
14. $\frac{10x-3y}{30}$.
15. $\frac{3a}{2b}$.
16. 0.
17. $\frac{2ac-4a^2+15bc}{12ac}$.
18. $\frac{x^4-y^4}{x^2y^2}$.
19. $\frac{1}{12c}$.
20. $\frac{22x-7}{105x}$.
21. 0.
22. $-3y$.

XXI. b. (p. 172).

1. $\frac{2x}{(x-1)(x+1)}$.
2. $\frac{2}{x-1}$.
3. $\frac{2x+7}{(x+3)(x+4)}$.
4. $\frac{1}{(x+3)(x+4)}$.
5. $\frac{9}{2x-3y}$.
6. $\frac{2x}{(x+6)(x+3)}$.
7. $\frac{11}{(2x-1)(2x+3)}$.
8. $\frac{x^2+y^2}{(x+y)(x-y)}$.
9. $\frac{3x}{(x+4)(x+10)}$.

10. $\dfrac{2x}{x-2}$. **11.** $\dfrac{12x}{(x-3)(x+3)}$. **12.** $\dfrac{7-3x}{(1-x)^2}$. **13.** $\dfrac{2(1-2x)}{(x+1)(x-1)}$.

14. $\dfrac{3x}{(x^2-y^2)}$. **15.** $\dfrac{-4y}{(x+y)^2}$. **16.** $\dfrac{1}{1-4x^2}$. **17.** $\dfrac{2b}{9a^2-4b^2}$.

18. $\dfrac{x}{(x-2y)^2}$. **19.** $\dfrac{1}{x+y}$. **20.** $\dfrac{7x}{x^2-16}$.

21. $\dfrac{3x+4y}{(x+y)(2x+3y)}$. **22.** $\dfrac{y}{(x-y)^2}$. **23.** 0.

24. $\dfrac{3a-3b}{c-d}$. **25.** $\dfrac{a^2+b^2}{ab(a-b)(a+b)}$. **26.** $\dfrac{4x-a}{a^2-4x^2}$.

27. $\dfrac{1}{6(a-b)}$. **28.** $\dfrac{x}{a}\left(\dfrac{a+x}{x-a}\right)$. **29.** $-\dfrac{3b}{a^2-9b^2}$.

30. $\dfrac{9b(a+3b)}{(a-2b)(2a+5b)}$. **31.** $\dfrac{a^2}{(a-1)(a^2+a+1)}$. **32.** $\dfrac{2xy}{x^3-8y^3}$.

33. $\dfrac{b}{27a^3+b^3}$. **34.** $5b$. **35.** $2x$.

36. 4. **37.** $\dfrac{4xy}{(x-y)^2}$. **38.** $\dfrac{2x}{(x-1)(x+1)}$.

39. $\dfrac{14}{(x-7)(x+7)}$. **40.** 0. **41.** $\dfrac{7y}{4}$.

XXI. c. (p. 173).

1. $\dfrac{4a}{a^2-b^2}$. **2.** $\dfrac{2b}{a^2-b^2}$. **3.** $\dfrac{3}{1-9x^2}$. **4.** $\dfrac{1}{(x-1)(x-3)(x-4)}$.

5. $\dfrac{a}{3(a^2-b^2)}$. **6.** $\dfrac{21-x}{6(x^2-9)}$. **7.** $\dfrac{2}{(x-1)(x-2)(x-3)}$.

8. $\dfrac{3a^2-ab}{a^3+b^3}$. **9.** $\dfrac{x}{x^3-27}$. **10.** $\dfrac{a^2}{(a-b)(a-c)}$.

11. $\dfrac{3}{(x-1)(x-3)}$. **12.** $\dfrac{3(x-y)}{(x-2y)(2x-y)}$. **13.** 0.

14. $\dfrac{7}{(x-1)(x-2)}$. **15.** $\dfrac{1}{2(x-4)}$. **16.** $\dfrac{4}{x+2y}$.

17. $\dfrac{3x}{(x-2)(x-3)(x+3)}$. **18.** $2(a+b)$. **19.** $\dfrac{2a^2}{a^3+b^3}$. **20.** 0.

21. $\dfrac{4}{(x^2-4)(x^2+4)}$. **22.** $\dfrac{2}{(x-1)(x+1)^2}$. **23.** $\dfrac{2}{x^2(x^2-4)}$.

24. $\dfrac{13}{(x-1)(x-2)(x-3)}$. **25.** $\dfrac{x^4+y^4}{x^6-y^6}$. **26.** $\dfrac{4}{(x^2-1)(x^2-4)}$.

27. $\dfrac{3y^2}{(x-2y)(x+3y)(x-3y)}$. **28.** $\dfrac{17x}{(x-7)(x+4)(x-3)}$.

29. $\dfrac{1-5x}{(x+3)(x+4)(x+7)}$. **30.** $\dfrac{3(a-3x^2)}{20(9a^2-x^4)}$. **31.** $\dfrac{5x^2+6x+13}{12(x-1)^2(x+1)^2}$.

32. $\dfrac{2(x^2+4xy+6y^2)}{(x+3y)(x+2y)}$. **33.** 0. **34.** $\dfrac{24x(7x+2)}{(9x^2+4)(9x^2-4)}$.

35. $\frac{2a^2b}{(a-b)(a-2b)(a-3b)}$.

36. $\frac{2}{(x+3y)(3x+y)}$.

37. $\frac{8x^2}{1-x^8}$.

38. $\frac{12}{(a^4-4)(a^4-1)}$.

39. $\frac{34xy}{y^2-16x^2}$.

40. $\frac{2x^2y^2}{x^4-y^4}$.

41. $\frac{4a^2}{a^2-b^2}$.

42. $3a-5b$.

43. $\frac{34xy}{49x^2-y^2}$.

44. $\frac{2xz}{(x-y-z)(x+y+z)}$.

45. $\frac{32a}{(a^2-9)(a^2-25)}$.

46. $\frac{1}{x-1}$.

47. $\frac{b^2}{(a+b)(a^2+b^2)}$.

48. $\frac{20x^2}{(3-2x)^3}$.

49. $\frac{16a}{1-a^4}$.

50. $\frac{3}{x^3-1}$.

51. $\frac{2}{x(x-2)}$.

52. $\frac{-4(x-2)}{x^4-1}$.

53. 0.

54. $\frac{3}{x(x^3-1)}$.

55. $\left(\frac{b}{a+b}\right)^3$.

56. $\frac{xy^2}{y^3-8x^3}$.

57. $\frac{-6}{(x+4)(x+3)(x+2)(x+1)}$.

58. $\frac{2(2x+5)(x^2+2)}{(x+1)^2(x^2-2x+3)}$.

59. $\frac{1}{(x-1)(x-2)(x-3)}$.

60. $\frac{2x}{x^2-xy+y^2}$.

61. $\frac{16x}{(x^2-1)(x^2-9)}$.

62. $\frac{1}{x^2-1}$.

63. $\frac{4x}{x^2-1}$.

64. $\frac{4+2a-a^2}{2a}$.

65. $\frac{3}{x(x+1)}$.

66. $\frac{3}{(x+1)(x+4)}$.

67. $\frac{a+bx}{b+ax}$.

68. $\frac{18x^2-18x+2}{(3x-2)(2x-1)(3x-4)}$.

69. $-\frac{3b}{(2a-3b)(a-4b)}$.

70. $\frac{1}{x}$.

71. $\frac{2xy}{x^2+y^2}$.

72. $-\frac{a}{b}$.

73. $\frac{x+1}{x-1}$.

74. $\frac{2xy}{x^2+y^2}$.

75. $\frac{6}{x^2}$.

76. $\frac{1}{x-2}$.

77. $\frac{x^3}{a^3}$.

78. 0.

79. $-a$.

80. 1.

81. a^2+b^2.

82. $-m$.

83. $\frac{1}{a^2}$.

84. $\frac{5(a+x)}{(2a-x)^2}$.

85. $\frac{3(a+2b)}{a-6b}$.

86. $\frac{5}{(x+1)(x+2)(x-3)}$.

87. $\frac{x(x+1)}{2}$.

88. $\frac{(x-5)^2}{(x-8)(3x-8)}$.

89. x^2+y^2.

90. $\frac{1}{x^2+y^2}$.

91. $\frac{(x-4)^2}{(x-7)(3x-5)}$.

92. $\frac{a^4-a^2b^2+b^4}{a^4+a^2b^2+b^4}$.

93. $\frac{3}{a}$.

94. $\frac{2}{x}$.

95. $\frac{27b^3}{8a^3+27b^3}$.

96. $\frac{x^2-ax+a^2}{x^2-a^2}$.

97. $\frac{2(a-b)x}{(a+x)(b+x)}$.

98. 3.

99. $2x$.

100. -1.

101. $\frac{1}{1+x}$.

102. $\frac{2xy}{x^2-y^2}$.

103. $\frac{2(ab+bc+ca)}{abc}$.

104. $(a+b)(c+a)$.

105. $\frac{(c+a)(c-a)}{(a+b)(a-b)}$.

106. $\frac{x^2+1}{x^2-1}$. 107. $-\frac{c}{a}$. 108. $2(x+y+z)$.

109. $\frac{3n-m}{2}$. 110. $\frac{x^2-3}{(x-1)^2}$. 111. 1. 112. $\frac{2}{x+y}$.

113. $\frac{2x(a+b)}{x^2-b^2}$. 114. 1. 115. 2. 116. 1.

117. $\frac{x(x+y+z)}{z(x-y+z)}$. 118. $y-x$. 119. 2. 120. $\frac{1}{x}$. 121. $1+a-a^2$.

XXII. (p. 180).

1. 8. 2. 3. 3. -2. 4. $4\frac{1}{5}$. 5. 2. 6. 1.
7. 7. 8. 1. 9. 2. 10. 3. 11. 12. 12. 2.
13. 7. 14. 3. 15. 7. 16. 7. 17. -107. 18. $\frac{5}{8}=\cdot63$.
19. $4\frac{2}{3}=4\cdot67$. 20. 16. 21. 6. 22. $\frac{9}{26}=\cdot35$. 23. 2. 24. $\frac{2}{5}=\cdot4$.
25. 2. 26. $\frac{5}{16}=\cdot31$. 27. $1\frac{1}{13}=1\cdot08$. 28. 4. 29. $1\frac{47}{120}=1\cdot39$.
30. -1. 31. 4. 32. $4\frac{5}{7}=4\cdot71$. 33. 0. 34. 5.
35. $\frac{1}{2}$. 36. 7. 37. $6\frac{1}{3}=6\cdot33$. 38. 6. 39. 2.
40. 5 41. 8. 42. $\frac{5}{7}=\cdot71$

XXIII. a. (p. 181).

1. $x(ax-b)$.
2. $(x+1)(x+10)$.
3. $3(x-1)(x+1)$.
4. $2(x-1)(x-3)$.
5. $(a-b)(x+a+b)$.
6. $(1-3x)(1+x)$.
7. $4(a-b)(a^2+ab+b^2)$.
8. $6(3x+1)(x+1)$.
9. $(4x-3)(2x+5)$.
10. $(x-1)(x+1)(x+2)$.
11. $5y(4x-3y)$.
12. $a(x-b)(x+b)$.
13. $(x-1)(x-51)$.
14. $(2a+1)(2a-1)$.
15. $(x+a)(x^2+a^2)$.
16. $(9+x)(8-x)$.
17. $(a+b)(a+b-1)$
18. $(2x-7)(8x+3)$.
19. $(a+b-c)(a-b+c)$.
20. $(ax-3)(bx-4)$.
21. $3(1-x)^2$.
22. $(3x-1)(9x-1)$.
23. $5(2a-3)(2a+3)$.
24. $(3a-2b)(x-y)$.
25. $3(a-3)(a^2+3a+9)$.
26. $(3-x^2)(2+x)$.
27. $(5x-4)(7x+8)$.
28. $(x-1)(x+1)(y-1)(y+1)$.
29. $(1-x)(2+x)(3-x)$.
30. $(x+y)(x-y)(a-b)(a^2+ab+b^2)$.
31. $7b(9a-3c-35b)$.
32. $(18x-y)(3x+y)$.
33. $(x+y)(6-a)$.
34. $\frac{1}{3}(3x-1)(3x+1)$
35. $(3x-2)(9x+4)$.
36. $7(7x-y)(7x+y)$.
37. $(x^2+1)(y-1)(y+1)$.
38. $(a-b)(a-b+1)(a-b-1)$.
39. $(x-2y)(x+2y)(x^2+2xy+4y^2)(x^2-2xy+4y^2)$.
40. $(a+b-2)(a+b-3)$.

41. $p(px-1)^2$. 42. $(x-12)(x-13)$.
43. $(x+4)(x+12)$. 44. $(11x-8y)(3x+4y)$.
45. $(x+2a)(x-7b)$. 46. $2b(3a^2+b^2)$.
47. $(3x-a)(5x+2b)$. 48. $2(x-2)(x+2)(x^2-2x+4)(x^2+2x+4)$.
49. $(x+1)(2x+1)(2x-3)$. 50. $(x^2+y^2)(a^2+b^2-c^2)$.

51. $(x-8)^2$. 52. $(a-b)(a+b+1)$.
53. $(x-7)(x+21)$. 54. $3(a-b)(a-b-1)$.
55. $(3x+2a)(4x-7b)$. 56. $(x+3)(x^2-x+1)$.
57. $(9x-5)(3x+25)$. 58. $(x-a)(x+a+3y)$.
59. $(a-2b+2c)(a+2b-2c)[a^2+4(b-c)^2]$. 60. $(\overline{a-1}\,.\,x+a)(ax-\overline{a+1})$.

61. $(a+b)(a+b+2)$. 62. $(5x-12y)(7x+2y)$.
63. $(x-y)(3x+3y-4)$. 64. $b^2(x-b)(x+b)(x^2+b^2)$.
65. $(x^2+y^2)^2$. 66. $(4x-a)(4x+a)$.
67. $32x(x+10)(x+1)$. 68. $2y(x+y)(x-y)$.
69. $(a+b-c)(a-b+c)(a+b+c)(b+c-a)$.
70. $3(a-b)(a+b)(5a^2-8ab+5b^2)$.

71. $(a-b)(5a+5b-1)$. 72. $(13x-4)(3x+2)$.
73. $(2x-1)(2x+1)(4x^2+1)$. 74. $(x-y)(a-b-c)$.
75. $(x-1)(x+1)(x-2)(x+2)$. 76. $(x+y-6a)(x+y-7a)$.
77. $16(a-b)(a+b)(5a^2-6ab+5b^2)$. 78. $(a-b)(ax+by+c)$.
79. $5x(13x^2+18xy+12y^2)$. 80. $(4x^2+2xy+y^2)(4x^2-2xy+y^2)$.

XXIII. b. (p. 182).

1. $a(x-a)(x+a)$, $(x+9y)(x-11y)$, $(75x-1)(x-1)$, $(x+y)(x-5)$.
2. $x-3$. 3. $\dfrac{2(4-x)}{(x-1)(x-2)(x-3)}$, 4. 4. $x^4-a^2x^2-b^2x^2+a^2b^2$.
5. $\pm 2·6$. $\pm 3·6$, $3·2$, $5·8$. 6. $\frac{1}{7}$. 7. 30 miles an hour.

XXIII. c. (p. 182).

1. $2(x-2)(x+2)$, $(2x-1)(x-2)$, $(a+b-c)(a+b+c)$, $(x-y)(x+y-3)$.
2. 1. 3. $12a^3b^3(a-b)$. 4. $3x-2$.
5. 22·4 acres. 6. $x=3$, $y=-6$. 7. 25 miles an hour.

XXIII. d. (p. 183).

1. $(2x+1)(x+3)$, $(a+b+x)(a-b-x)$, $(b-c)(a-c)$, $3(1-b)(1+b+b^2)$.
2. $x-a$. 3. 0. 4. $\left.\begin{array}{l} x=6,\ 4,\ 2, \\ y=1,\ 2,\ 3. \end{array}\right\}$
5. $a^4+a^3b-ab^3-b^4$. 6. 5. 7. 2 stumped, 8 caught, 5 bowled.

XXIII. e. (p. 183)

1. $(x-32)(x+4)$, $(x+y)(a-2)$, $(x-1)^2(x-3)$, $4(1+3a)(1-3a+9a^2)$.

2. $\dfrac{c-a+b}{c+a-b}$. 3. $(x+1)(x-2)(x-3)$. 4. 25·7 miles from the start.

5. x^2-2x+3. 6. -15. 7. $2\frac{1}{2}$.

XXIII. f. (p. 184).

1. $(2x-1)(x+5)$, $3(a-b)(a+b)$, $(b+c)(a-d)$, $(x-y)(x+y)(x-z)$.

2. $a-b+c$. 3. $\dfrac{3-2x^2}{(1-x)^2(2-x)^2}$. 4. $\left.\begin{matrix}x=9, 6, 3, 0,\\ y=1, 3, 5, 7.\end{matrix}\right\}$

5. $x=4$, $y=3$. 6. 15 miles. 7. $-7\frac{1}{2}$.

XXIII. g. (p. 184).

1. $(3x+4)(4x-3)$, $(2a+b+c-d)(2a+b-c+d)$, $(x-1)(x+1)(x+2)$, $(x-1)(x+1)(y-1)(y+1)$.

2. $\dfrac{1}{x^2-1}$. 3. $18x^2y^2(x^4-y^4)$. 4. 184 against, 161 for.

5. $x^2-x(a+2b)+3b^2+a^2$. 6. -3. 7. $\dfrac{5280}{x}$ min., $20x$ yds., $\dfrac{xy}{88}$ miles.

XXIII. h. (p. 184).

1. $6x+\dfrac{2}{x^3}$. 2. 0. 3. 3·3, 4·8. 4. 1.

5. $\frac{5}{7}$. 6. $x=-2$, $y=1\frac{1}{3}$. 7. £$3x$, £$12x$, £$\dfrac{ax}{100}$, £$\dfrac{axy}{100}$.

XXIII. k. (p. 185).

1. $x+1+\dfrac{1}{x}$. 2. 2. 3. $\frac{4}{5}$.

4. $\dfrac{xy^2}{x^2+xy+y^2}$. 5. 22 min past 4. 6. $x=-1$, $y=-11$.

7. -15, -8, -3, 0, 1, 0, -3, -8, -15.

XXIII. l. (p. 185).

1. $6xy-3y^2$. 2. 3. 3. $\dfrac{a+b-c}{a-b-c}$.

4. 31, 4. 5. $a=9\frac{1}{2}$, $b=4$. 6. $x=-2$, $y=-2$. $x=-\frac{1}{2}$, $y=-\frac{1}{2}$.

XXIII. m. (p. 186).

1. $12ab$. 2. $2\frac{1}{2}$. 3. $\frac{3}{7}$.

4. $\dfrac{4(x^2+x+1)(x+1)}{x^4(x^4+1)}$. 5. 55 min. past 4.

6. The equation is an identity. 7. £$\left(85+\dfrac{17x}{20}\right)$, £$\dfrac{9200}{100+x}$.

XXIV. a. (p. 187).

1. x^4. 2. a^5. 3. y^6. 4. x^3y^2. 5. ab^2. 6. x^4y^3. 7. $2ab$.
8. $4a^2b$. 9. $7x^2y^3z^4$. 10. $\frac{2a}{b}$. 11. $\frac{3x^2}{y^3}$. 12. $\frac{9a^2b^3}{c^4}$.
13. ·1. 14. ·5. 15. ·8. 16. 100. 17. $\frac{5}{2}$.
18. $\frac{7}{6}$. 19. $\frac{b^2c}{10}$. 20. $\frac{a}{5b^2}$. 21. $\frac{11a^3c^5}{10}$. 22. $\frac{4x^6y^8}{7}$.
23. $\frac{10a^2}{9b}$. 24. $\frac{8x^2}{y^6}$. 25. $3(a-b)$. 26. $\frac{11}{3}(2x+y)$. 27. $x+y$.

XXIV. b. (p. 188).

1. $x+y$. 2. $x-y$. 3. $a+2b$. 4. $2a-b$.
5. $x-3$. 6. $1-2x$. 7. $5a-3b$. 8. $7x-y$.
9. $2a-7b$. 10. $3x+4y$. 11. $11a-2b$. 12. $1-x^3$.
13. $13a+2b$. 14. $9a-b$. 15. $5x-7y$. 16. a^2-b^2.
17. $2a^2+b^2$. 18. x^2y-1. 19. $\frac{x}{3}-1$. 20. a^2+2b^2.
21. $x-\frac{1}{2}$. 22. $\frac{a}{2}-b$. 23. $\frac{x}{y}-\frac{y}{x}$. 24. $x-\frac{3y}{2}$.
25. $x^2+\frac{1}{x^2}$. 26. $a-\frac{5}{2}$. 27. $x+y+1$. 28. $2b$.
29. $x-y-2$. 30. $3(a+b)+1$. 31. $a+b+c+d$. 32. $2a+b$.
33. $\frac{a}{b}$. 34. $4(x-y)-1$. 35. $a+2b+\frac{1}{2}$. 36. b.
37. $\frac{a}{b}-2$. 38. $x+7y$. 39. $\frac{a^3}{x^3}-\frac{x^3}{a^3}$. 40. $\frac{2a^2}{x^2}-\frac{x^2}{a^2}$.
41. $\frac{x^4}{2a^4}+\frac{2a^4}{x^4}$. 42. $\frac{a+b}{3}-\frac{x+y}{2}$. 43. $\pm 2ab$. 44. 4.
45. $\pm 6x$. 46. $\pm 20xy$. 47. 1. 48. ± 2. 49. 1. 50. $\pm\frac{q}{3}$.

XXIV. c. (p. 191).

1. x^2+x+1. 2. $2x^3+x+1$. 3. x^2-x+2.
4. $a^2-2ab+b^2$. 5. $3x^2-2x+5$. 6. $2x-5y+4z$.
7. $x(4x^2+3x+1)$. 8. $5x^2-2ax-3a^2$. 9. $x^2-3+\frac{1}{x^2}$.
10. $a-b-c$. 11. x^3-3x-7. 12. $3x^2-2xy+5y^2$.
13. $a-2b+3c$. 14. $3a^2-7b^2-11c^2$. 15. $2ab-3bc-ca$.
16. $2x-3y+5z$. 17. $7x^2-5xy+6y^2$. 18. $x^3-2-\frac{1}{x^3}$.
19. $2x^2-3y^2+7z^2$. 20. $\frac{x}{y}-1+\frac{y}{x}$. 21. $\frac{a^2}{2}-a-1$.
22. $\frac{a^2}{3}+a+\frac{1}{2}$. 23. $\frac{3a^2}{5}+\frac{2a}{3}+1$. 24. $\frac{b^2}{3}-\frac{a}{2}+1$.

25. $x^3-\frac{x^2}{2}+\frac{1}{3}$. **26.** $\frac{x^2}{2}-3x+\frac{1}{3}$. **27.** $\frac{x^2}{3}-2x+\frac{a}{2}$.

28. $3x^2+4-\frac{8}{x^2}$. **29.** $\frac{2x}{y}-\frac{1}{4}-\frac{3y}{2x}$. **30.** $a^2-\frac{3a}{4}+\frac{4}{5}$.

XXIV. d. (p. 193).

1. 42. **2.** 135. **3.** 130. **4.** 52. **5.** 187. **6.** 625.
7. 462. **8.** 84. **9.** 126. **10.** 2005. **11.** 3001.
12. 1973. **13.** 2345. **14.** 20202. **15.** 1351. **16.** 3489.

XXIV. e. (p. 201).

18. 7·32, 7·60, 7·71, 7·85.
19. 6·21, 6·30, 6·33, 6·53, 6·84. 41·5, 44·6, 46·5.
20. 7·06, 7·12, 7·16, 7·34. 49·7, 51·3, 53, 54·2.
21. 7·39, 7·67, 7·90. **22.** 80·2, 81·7. **23.** 80·15, 80·23, 80·54.
24. 9·053, 9·088. **25.** 91·35, 91·78. **26.** 10·084, 10·048.
27. 12·36, 12·94. **28.** 1·73. **29.** 2·45.
30. 2·65. **31.** 3·32. **32.** 2·37. **33.** 2·19.
34. 2·57. **35.** 2·12. **36.** 2·39. **37.** 2·07.
38. 5·24, 5·83. **39.** 2·47, 2·76. **40.** 2·02. **41.** 3·03.
42. 3·06. **43.** 3·08. **44.** 3·11. **45.** 4·03.
46. 4·08. **47.** 4·09. **48.** 5·03. **49.** 5·05. **50.** 5·07.

XXV. a. (p. 203).

1. 1, 2. **2.** 1, −1. **3.** a, b. **4.** 0, 1.
5. −2, −3. **6.** $-a$, b. **7.** 0, −2. **8.** $2a$, b.
9. $-a$, $2b$. **10.** $\frac{1}{2}$, $-\frac{3}{4}$. **11.** $-\frac{1}{5}$, $-\frac{3}{8}$. **12.** 0, $-\frac{1}{3}$.
13. $\frac{a}{2}$, $\frac{b}{3}$. **14.** $a+b$, $a-b$. **15.** $\frac{a+b}{2}$, $-\frac{c+d}{2}$.
16. $p-2q$, $2p-q$. **17.** $2(a+b)$, $-3(a-b)$.
18. a^2, $-b^2$. **19.** $-(a-b)^2$, $(a+b)^2$. **20.** 3.
21. 0, a. **22.** 0, −4. **23.** $-a$. **24.** $-2a$.

XXV. b. (p. 205).

1. 5, 2. **2.** 3, 2. **3.** ±2. **4.** 0, 3. **5.** $-1, -$
6. −5, +1. **7.** 1, 7. **8.** 2, −1. **9.** ±2. **10.** 10, 1.
11. −9, 5. **12.** 3, 9. **13.** −5, 4. **14.** 7, 0. **15.** ±1.
16. 2, 2. **17.** −3, 0. **18.** −7, −3. **19.** 15, −1.
20. 5, −8. **21.** 15, 15. **22.** ±3. **23.** 0, 2.
24. 0, −7. **25.** −102, 1. **26.** −1, −15.

XXV. c. (p. 207).

1. $1\frac{1}{2}, 4$.
2. $-\frac{1}{3}, \frac{1}{2}$.
3. $-1\frac{1}{3}, -1\frac{1}{5}$.
4. $0, -1\frac{2}{7}$.
5. $1\frac{2}{5}, -\frac{1}{5}$.
6. $1\frac{1}{7}$.
7. $\frac{a}{2}, \frac{b}{2}$.
8. $-\frac{a}{5}, -\frac{b}{6}$.
9. $\frac{a+b}{2}, \frac{c+d}{3}$.
10. $-1\frac{1}{4}, 4\frac{1}{2}$.
11. $1, -2$.
12. $5, 3$.
13. $-4, 8$.
14. $4, 6$.
15. $5, -1$.
16. $\pm\frac{1}{2}$.
17. $4, 4$.
18. $1, \frac{1}{2}$.
19. $-4, 6$.
20. $0, -3\frac{2}{5}$.
21. $10, 1$.
22. $-\frac{1}{2}, -\frac{1}{2}$.
23. $-4{\cdot}1, -{\cdot}7$.
24. $1, 1$.
25. $4, \frac{1}{2}$.
26. $\frac{3}{2}, -\frac{4}{5}$.
27. $\frac{3}{4}, -4$.
28. $\frac{3}{2}, -\frac{7}{5}$.
29. $2, -1$.
30. $-9\frac{1}{2}, 1$.
31. $15, -4$.
32. $2, -\frac{1}{150}$.
33. $\frac{2}{3}, -\frac{2}{5}$.
34. $-\frac{5}{4}, -\frac{7}{6}$.
35. $1, -\frac{7}{13}$.
36. $\frac{5}{7}, -\frac{7}{10}$.
37. $-\frac{2}{3}, \frac{8}{3}$.
38. $11, -13$.

XXV. d. (p. 211).

1. $\frac{1}{2}, -\frac{2}{3}$.
2. $\frac{1}{13}, -\frac{1}{2}$.
3. $\frac{1}{12}, -\frac{1}{13}$.
4. $1, -\frac{1}{5}$.
5. $\frac{2}{3}, 5$.
6. $-5, \frac{3}{7}$.
7. $-9, -\frac{1}{2}$.
8. $5, -3$.
9. $2, \frac{1}{2}$.
10. $\frac{3}{2}, \frac{1}{3}$.
11. $-\frac{1}{3}, 3$.
12. $\frac{4}{3}, -\frac{3}{4}$.
13. $\frac{5}{6}, -\frac{3}{2}$.
14. $3, -2$.
15. $\frac{5}{2}, -\frac{13}{6}$.
16. $\frac{9}{5}, -\frac{4}{3}$.
17. $2, -\frac{43}{25}$.
18. $\frac{11}{5}, 1$.
19. $\frac{9}{5}, -\frac{1}{2}$.
20. $22, -2$.
21. $-\frac{4}{3}, -\frac{3}{5}$.
22. $\frac{3}{2}, 4$.
23. $1, -\frac{1}{2}$.
24. $\frac{3}{2}, -\frac{10}{3}$.
25. $2, -3$.
26. $2, -14$.
27. $5, \frac{18}{7}$.
28. $5, -\frac{3}{2}$.
29. $0, 7\frac{10}{23}$.
30. $12, 36$.
31. $0, 3\frac{1}{2}$.
32. $3, -2\frac{1}{3}$.
33. $\frac{3}{2}, 4$.
34. $4, -\frac{9}{4}$.

XXV. e. (p. 212).

1. $1\pm\sqrt{2}=2{\cdot}41$ or $-{\cdot}41$.
2. $-1\pm\sqrt{3}={\cdot}73$ or $-2{\cdot}73$.
3. $2\pm\sqrt{3}=3{\cdot}73$ or ${\cdot}27$.
4. $1\pm\sqrt{5}=3{\cdot}24$ or $-1{\cdot}24$.
5. $\frac{9\pm\sqrt{161}}{10}=2{\cdot}17$ or $-{\cdot}37$.
6. $1\pm\sqrt{6}=3{\cdot}45$ or $-1{\cdot}45$.
7. $\sqrt{3}=1{\cdot}73$.
8. $-6\pm\sqrt{3}=-7{\cdot}73$ or $-4{\cdot}27$.
9. $\frac{6\pm\sqrt{176}}{10}=1{\cdot}93$ or $-{\cdot}73$.
10. $2\pm\sqrt{13}=5{\cdot}61$ or $-1{\cdot}61$.
11. $\frac{-5\pm\sqrt{73}}{4}=-3{\cdot}39$ or ${\cdot}89$.
12. $\frac{9\pm\sqrt{3}}{3}=3{\cdot}58$ or $2{\cdot}42$.
13. $\frac{1\pm\sqrt{2}}{3}={\cdot}80$ or $-{\cdot}14$.
14. $2\sqrt{3}=3{\cdot}46$ or $-\sqrt{3}=-1{\cdot}73$.

XXV. f. (p. 214).

1. $\pm 5, \pm 2$. 2. $\pm 3, \pm 6$. 3. $1, 3$. 4. $0, -1$.
5. $3, -1, 1 \pm \sqrt{13} = 4{\cdot}61$ or $-2{\cdot}61$. 6. $1, -1, -1$. 7. $\pm 1, \frac{4}{3}$.
8. $5, -1, 2 \pm \sqrt{3} = 3{\cdot}73$ or ${\cdot}27$. 9. $\pm 2, \pm \frac{1}{2}, \pm 1$.
10. $-8, 3, 0, -5$. 11. $0, 5, -6$.
12. $0, -5$, (other roots imaginary). 13. $0, -\frac{5}{2}$.
14. $-5, 2$, (other roots imaginary).
15. $1, -4, \dfrac{-3 \pm \sqrt{11}}{2} = {\cdot}16$ or $-3{\cdot}16$.
16. $-\frac{3}{2}, \dfrac{\pm \sqrt{10} - 3}{2} = -3{\cdot}08, {\cdot}08$. 17. $1, 2, \dfrac{-5 \pm \sqrt{17}}{2} = -4{\cdot}56, -{\cdot}44$.

XXVI. (p. 219).

7. $2{\cdot}5, -1{\cdot}5$. 8. $-2{\cdot}5, 3{\cdot}5$. 9. ${\cdot}5, -1{\cdot}6$.
10. ${\cdot}8, 2{\cdot}5$. 11. $1{\cdot}5, 2{\cdot}3$. 12. ${\cdot}5, -2{\cdot}6$.
13. $2{\cdot}1, -1{\cdot}5$. 14. The roots are equal, ${\cdot}5$.
15. The roots are imaginary. 17. $3{\cdot}8, -{\cdot}8$. 18. $-2, 2{\cdot}6$.
19. $-2, 3{\cdot}5$. 20. $-3, 4{\cdot}6$.
21. $1{\cdot}87, -1{\cdot}07$. Minimum value $-10{\cdot}8$. 22. $-2, 3$.
23. $4, -2{\cdot}5$. 24. $4{\cdot}8, {\cdot}2$.
25. $-1, 2{\cdot}2, 3, 3{\cdot}4, 3{\cdot}4, 3$. Maximum value $3{\cdot}45$.
26. $(3, 5)$. 27. $1{\cdot}44$. 28. 6. 29. $2{\cdot}5\ 2{\cdot}5$.
30. $2{\cdot}6, 1$. 31. -4. 32. $-1{\cdot}4, 2{\cdot}6$. 33. $2{\cdot}5, -4$.

XXVII. a. (p. 222).

1. $x=3, y=1$. 2. $x=5, y=-2$. 3. $x=2, y=8$.
4. $x=7, y=2$. 5. $x=3, y=5$. 6. $x=1, y=2$.
7. $x=2, y=-1$. 8. $x=6, y=-3$. 9. $x=5, y=2$.
10. $x=6, 9$. $y=9, 6$. 11. $x=5, -3$. $y=3, -5$. 12. $x=12, -11$. $y=11, -12$.
13. $x=13, -9$. $y=-9, 13$. 14. $x=-7, 13$. $y=13, -7$. 15. $x=7, -3$. $y=3, -7$.
16. $x=\frac{1}{2}, \frac{1}{4}$. $y=\frac{1}{4}, \frac{1}{2}$. 17. $x=2, \frac{3}{4}$. $y=3, 8$. 18. $x=2, -\frac{1}{5}$. $y=1, -10$.
19. $x=6, -\frac{4}{3}$. $y=2, -9$. 20. $x=4, 1{\cdot}6$. $y=2, 5$. 21. $x=\pm 7, \pm 2$. $y=\pm 2, \pm 7$.
22. $x=\pm 5, \pm 3$. $y=\mp 3, \mp 5$. 23. $x=\pm 2, \pm \frac{1}{2}$. $y=\pm 1, \pm 4$. 24. $x=\pm 3$. $y=\pm 1$.
25. $x=\pm 2, \pm \frac{10}{3}$. $y=\pm 5, \pm 3$. 26. $x=\pm 5, \pm 3$. $y=\pm 2{\cdot}4, \pm 4$. 27. $x=4, 2$. $y=2, 4$.

28. $x=\frac{1}{2}, -\frac{1}{3}$. $y=\frac{1}{3}, -\frac{1}{2}$.

29. $x=5, 9$. $y=9, 5$.

30. $x=7, -5$. $y=5, -7$.

31. $x=1, -2$. $y=-1, \frac{1}{2}$.

32. $x=\frac{1}{2}$. $y=\frac{1}{3}$.

33. $x=5, 1\frac{1}{2}$. $y=-2, -6\frac{2}{3}$.

34. $x=2, 1\frac{2}{5}$. $y=1, 1\frac{3}{7}$.

35. $x=7, -2$. $y=-2, 7$.

36. $x=\frac{1}{2}, -\frac{1}{4}$. $y=\frac{1}{4}, -\frac{1}{2}$.

37. $x=5, 1$. $y=2, 10$.

38. $x=3, 0$. $y=0, -9$.

39. $x=5, 11$. $y=11, 5$.

40. $x=13, -12$. $y=12, -13$.

41. $x=2, 4$. $y=2, 1$.

42. $x=3, 1\frac{1}{3}$. $y=-2, -4\frac{1}{2}$.

XXVII. b. (p. 224).

1. $x=1, 2$. $y=2, 1$.

2. $x=4, -3$. $y=3, -4$.

3. $x=3, 2$. $y=4, 6$.

4. $x=5, 4$. $y=-2, -2\frac{1}{2}$.

5. $x=1, \frac{2}{3}$. $y=1, \frac{3}{2}$.

6. $x=4, -1\frac{1}{2}$. $y=1, -2\frac{2}{3}$.

7. $x=\pm 1, \pm 2$. $y=\pm 2, \pm 1$.

8. $x=\pm 5, \pm 4$. $y=\pm 4, \pm 5$.

9. $x=\pm 4, \pm 3$. $y=\pm 3, \pm 4$.

10. $x=\pm 7 \pm 2$. $y=\pm 2 \pm 7$.

11. $x=\frac{1}{2}, \frac{1}{3}$. $y=\frac{1}{3}, \frac{1}{2}$.

12. $x=\frac{1}{4}, -\frac{1}{5}$. $y=\frac{1}{5}, -\frac{1}{4}$.

13. $x=\frac{1}{5}, \frac{2}{5}$. $y=\frac{1}{3}, \frac{1}{10}$.

14. $x=3, -15$. $y=5, -1$.

15. $x=\pm\frac{1}{5}, \pm\frac{1}{6}$. $y=\pm\frac{1}{6}, \pm\frac{1}{5}$.

16. $x=\pm\frac{1}{2}, \pm 1$. $y=\pm 2, \pm 1$.

17. $x=\frac{1}{5}, -\frac{1}{3}$. $y=\frac{1}{3}, -\frac{1}{5}$.

18. $x=4, \frac{1}{4}$. $y=\frac{1}{4}, 4$.

19. $x=2, -\frac{1}{2}$. $y=\frac{1}{2}, -2$.

20. $x=8, 2$. $y=4, 16$.

21. $x=\frac{1}{2}, \frac{1}{3}$. $y=\frac{1}{3}, \frac{1}{2}$.

22. $x=\frac{1}{5}, -\frac{1}{4}$. $y=\frac{1}{4}, -\frac{1}{5}$.

23. $x=2, 7$. $y=7, 2$.

24. $x=9, -3$. $y=3, -9$.

25. $x=\frac{1}{2}$. $y=1$.

26. $x=\frac{1}{2}$. $y=\frac{1}{3}$.

XXVII. c. (p. 226).

1. $x=\pm 1$. $y=\pm 2$.

2. $x=\pm 3, \pm\sqrt{2}$. $y=\pm 2, \mp 4\sqrt{2}$.

3. $x=\pm 3, x=\mp 1$. $y=\pm 2, y=\pm 2$.

4. $x=\pm 1$ (other roots imaginary). $y=\pm 1$.

5. $x=\frac{3}{5}, 1$. $y=\frac{14}{5}, 2$.

6. $x=\pm 3, 0$. $y=\pm 1, \pm 2$.

7. $x=\pm\frac{3}{\sqrt{7}}$. $y=\pm\frac{4}{\sqrt{7}}$.

8. $x=\pm 10$. $y=\pm 2$.

9. $x=\pm\frac{5}{\sqrt{6}}$. $y=\pm\frac{1}{\sqrt{6}}$. **10.** $x=\pm 3,\ \pm\frac{5}{\sqrt{2}}$. $y=\pm 2,\ \pm\frac{1}{\sqrt{2}}$. **11.** $x=\pm 2$. $y=\pm 3$.

12. $x=\pm 2$. $y=\pm 3$. **13.** $\left.\begin{array}{l} x=4,\ 2. \\ y=2,\ 4. \end{array}\right\}$ other roots imaginary.

14. $x=\pm 2$. $y=\mp\frac{1}{3}$. **15.** $x=\pm 2,\ \pm\frac{3}{\sqrt{2}}$. $y=\pm 1,\ \pm\frac{1}{\sqrt{2}}$. **16.** $x=\pm 7$. $y=\pm 5$.

17. $x=8,\ -3$. $y=3,\ -8$. **18.** $x=-7,\ 3,\ 5,\ -1$. $y=7,\ -3,\ 1,\ -5$. **19.** $x=4,\ -6\frac{2}{9}$. $y=6,\ -4\frac{2}{7}$.

20. $x=\pm 5,\ \pm 2$. $y=\mp 4,\ \mp 3$. **21.** $x=\pm 2$. $y=\pm 1$. **22.** $x=5,\ 4$. $y=4,\ 5$.

23. $x=1,\ -3\frac{1}{2}$. $y=1,\ -\frac{2}{7}$. **24.** $x=-7,\ 4$. $y=-\frac{21}{4},\ 3$. **25.** $x=\frac{2}{3},\ -\frac{1}{3}$. $y=\frac{3}{2},\ 0$.

26. $x=\pm 2,\ \pm\frac{7}{\sqrt{2}}$. $y=\pm 5,\ \mp\frac{3}{\sqrt{2}}$. **27.** $x=7,\ -\frac{19}{4}$. $y=3,\ -\frac{23}{8}$. **28.** $x=\pm 5,\ \pm 3$. $y=\pm 3,\ \pm 5$.

29. $x=\pm 3,\ \pm\frac{8}{\sqrt{6}}$. $y=\pm 1,\ \pm\frac{1}{\sqrt{6}}$. **30.** $x=2\frac{1}{2},\ -1\frac{3}{4}$. $y=-1\frac{1}{6},\ 1\frac{2}{3}$. **31.** $x=2,\ 5,\ 1\pm\sqrt{6}$. $y=-5,\ -2,\ -1\pm\sqrt{6}$.

32. $x=\pm 3,\ \pm 1$. $y=\mp 1,\ \mp 3$. **33.** $x=2,\ -3,\ -2\pm\sqrt{2}$. $y=3,\ 3,\ -1\pm\sqrt{2}$.

34. $\left.\begin{array}{l} x=0,\ -2. \\ y=-4,\ 2. \end{array}\right\}$ other roots imaginary. **35.** $x=\pm 3,\ \pm 2,\ \pm 3,\ \pm 2$. $y=\pm 2,\ \pm 3,\ \mp 2,\ \mp 3$.

XXVII. d. (p. 229).

1. A circle, centre (0, 0), radius 6. **2.** The origin.
3. ,, ,, ,, 7. **4.** A circle, centre (0, 0), radius 9.
5. A circle through the origin, centre $(-4, 4)$, radius $4\sqrt{2}$.
6. ,, ,, ,, ,, (4, 3), ,, 5.
7. A circle, centre (3, 4), radius 6. **8.** A circle, centre (1, 2), radius 6.
9. ,, ,, $(-2, 3)$, ,, 5. **10.** ,, ,, $(3, -3)$, ,, 4.
11. ,, ,, $(-1, 0)$, ,, 4. **12.** ,, ,, (2, 0), ,, 5.
13. ,, ,, (1, 0), ,, 4. **14.** ,, ,, (7, 0), ,, 6.
15. A circle, centre (0, 0), radius $\sqrt{2}$.
16. ,, ,, (0, 0), ,, $\sqrt{5}$.
17. ,, ,, (0, 0), ,, $\sqrt{13}$.

18. A circle, centre (0, 0), radius $\sqrt{10}$.
19. " " (0, 0), " $2\sqrt{5}$.
20. " " (0, 0), " $\sqrt{3}$.
21. " " (−1, −1), " $\sqrt{2}$, through the origin.
22. " " (1, 0), " $\sqrt{2}$.
23. " " (−2, 2), " $\sqrt{5}$.
24. " " (−1, −1), " $\sqrt{5}$.
25. " " (3, −2), " $\sqrt{10}$.
26. " " (0, 0), " $\frac{\sqrt{10}}{2}$.
27. " " (1, −2), " $\sqrt{3 \cdot 5}$.
28. " " (2, −1), " 1·5.
29. " " (3, 0), " 2·5.

XXVII. e. (p. 233).

1. $x=5{\cdot}3,\ 1{\cdot}7.$ $y=1{\cdot}7,\ 5{\cdot}3.$
2. $x=6{\cdot}56,\ 2{\cdot}44.$ $y=2{\cdot}44,\ 6{\cdot}56.$
3. $x=5{\cdot}1,\ -3{\cdot}1.$ $y=3{\cdot}1,\ -5{\cdot}1.$
4. $x=5{\cdot}61,\ -1{\cdot}61.$ $y=1{\cdot}61,\ -5{\cdot}61.$
5. $x=6{\cdot}19,\ {\cdot}81.$ $y={\cdot}81,\ 6{\cdot}19.$
6. $x=4{\cdot}7,\ -1{\cdot}7.$ $y=1{\cdot}7,\ -4{\cdot}7.$
7. 4, 9.
8. 3·2, 7·8.
9. 5·73, 2·27.
10. 5·12, −3·12.
11. $x=1{\cdot}27,\ -{\cdot}47.$ $y=1{\cdot}54,\ -1{\cdot}94.$
12. $x=-2,\ 2{\cdot}8.$ $y=2,\ -{\cdot}4.$
13. $x=2{\cdot}6,\ -{\cdot}2.$ $y=3{\cdot}2,\ {\cdot}4.$
14. $x=1,\ -2{\cdot}2.$ $y={\cdot}5,\ 2{\cdot}1.$
15. $x={\cdot}69,\ -2{\cdot}61.$ $y=-2{\cdot}92,\ 1{\cdot}48.$
16. $x=\pm 5{\cdot}29,\ \pm 2{\cdot}84.$ $y=\pm 2{\cdot}84,\ \pm 5{\cdot}29.$
17. $x=\pm 13{\cdot}8,\ \pm 5{\cdot}8.$ $y=\pm 5{\cdot}8,\ \pm 13{\cdot}8.$
18. $x=9{\cdot}3,\ -4{\cdot}3.$ $y=8{\cdot}6,\ -18{\cdot}6.$

XXVIII. (p. 234).

1. (i) $x+y$ miles, (ii) $x-y$ miles, (iii) $\frac{a}{x+y}$ hours, (iv) $\frac{a}{x-y}$ hours.
2. (i) £$\frac{x}{100}$, (ii) £$\frac{xy}{100}$, (iii) £$\frac{xyz}{100}$, (iv) £$\left(z+\frac{xyz}{100}\right)$.
3. (i) £$\frac{10000}{100+x}$, (ii) £$\frac{100a}{100+x}$, (iii) £$\frac{10000}{100+xy}$, (iv) £$\frac{100a}{100+xy}$.
4. (i) $\frac{1}{y}$ hours, (ii) $\frac{z}{y}$ hours, (iii) $\frac{3z}{2y}$ hours, (iv) ay miles.
5. (i) $\frac{x+y}{xy}$, (ii) $\frac{a(x+y)}{xy}$, (iii) $\frac{xy}{x+y}$ hours, (iv) $\frac{3xy}{4(x+y)}$ hours.

6. (i) $\frac{yz+zx-xy}{xyz}$, (ii) $\frac{xyz}{yz+xz-xy}$ hours.

7. (i) £$\frac{x}{z}$, (ii) £$\frac{x}{yz}$, (iii) £$\frac{100x}{yz}$, (iv) £$\frac{abx}{yz}$.

8. (i) £$(z-y)$, (ii) £$\left(\frac{z-y}{x}\right)$, (iii) £$\frac{z-y}{xy}$, (iv) £$\frac{ab(z-y)}{xy}$,
(v) $\frac{100(z-y)}{xy}$ per cent.

9. (i) $\frac{x}{12}$ pence, (ii) $\frac{xy}{12}$ pence, (iii) $\frac{x+1}{12}$ pence,
(iv) $\frac{(x+1)y}{12}$ pence, (v) $\frac{ax}{12}$ pence, (vi) $\frac{a(x+1)}{12}$ pence.

10. (i) £$\frac{x}{100}$, (ii) £$\left(1+\frac{x}{100}\right)$, (iii) £$a\left(1+\frac{x}{100}\right)$,
(iv) £$\frac{ax}{100}$, (v) £$\left(1+\frac{x}{100}\right)^2$, (vi) £$\left(1+\frac{x}{100}\right)^3$,
(vii) £$\left(1+\frac{x}{100}\right)^n$, (viii) £$P\left(1+\frac{x}{100}\right)^2$, (ix) £$P\left(1+\frac{x}{100}\right)^3$,
(x) £$P\left(1+\frac{x}{100}\right)^n$, (xi) £$\left\{P\left(1+\frac{x}{100}\right)^n-P\right\}$.

11. (i) £$\frac{100x}{100+x}$, (ii) £$\frac{ax}{100+x}$, (iii) £$\frac{100xy}{100+xy}$, (iv) £$\frac{axy}{100+xy}$.

12. (i) $\frac{7}{4x}$, (ii) $\frac{9}{2x}$.

13. (i) $(x-y)$ miles, (ii) $a(x-y)$ miles, (iii) $\frac{1}{x-y}$ hours, (iv) $\frac{b}{x-y}$ hours.

14. $(x+2)(x+3)-x(x+1)=y$. 15. $ax+by=\frac{z}{20}$. 16. $\frac{ax}{12}+\frac{by}{10}=12z$

17. $y^2-(y-8)^2=x$. 18. $z^2-(z-2y)^2=a$. 19. $\frac{x+b}{y-c}-\frac{x}{y}=a$.

20. $ax+by=(x+y)c$. 21. $(x+a)(y+a)=2xy$. 22. $\frac{3a}{x}-\frac{3a}{y}=n$.

23. $\frac{xy}{x+y}=z$. 24. $ax=y(a-n)$. 25. $ax+ay=n$. 26. $ax+(a-b)y=n$

27. $(x-1)y=1760$. 28. $(x-1)y=1760a$. 29. $\frac{x}{3}+\frac{x}{5}+\frac{x}{10}+y=x$, or $11x=30y$

30. $y+\frac{xy}{100}=z$. 31. $\frac{ax}{100}-\frac{by}{100}=c$. 32. $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=d$. 33. $ay-z(x-a)=20$

XXIX. a. (p. 239).

1. 10, 12. 2. 16 ft., 12 ft. 3. 16, 18. 4. 15, 16. 5. $5\frac{1}{2}$. 6. 7
7. 169. 8. 53 yds., 106 yds. 9. 3 ft. 10. 12, 15. 11. 3 ft. 12. 5, 8
13. 15. 14. 12. 15. 6, 9. 16. 72. 17. 40 yds., 50 yds. 18. 4. 19. 2
20. 30. 21. 10. 22. 11. 23. 55 and 60 miles per hr. 24. 12 and 24 days
25. 2 hours, 4 hours. 26. 25 miles per hr. 27. $\frac{2}{5}$. 28. 6, 7, 8, 9, 10.
29. 95. 30. 4 feet. 31. 32 miles per hr. 32. 4.

XXIX. b. (p. 239B).

1. 5, 7. 2. 3 in. 3. 43. 4. 12. 5. 93.
6. 6 yds. per sec. 7. 14, 11. 8. 6 miles an hour. 9. 7.
10. 55, 60 miles an hour. 11. 6s. 6d. 12. 13 miles. 13. 32.
14. 24 ft. long, 18 ft. wide, 11 ft. high.
15. 10 yds., 7 yds. square, £7, £5.
16. 30 miles an hour, 50 miles an hour.
17. 14 ft. long, 12 ft. wide, 9 ft. high. 18. 8 miles an hour.
19. 5 miles an hour. 20. 8 ft., $7\frac{1}{2}$ ft. 21. 576.
22. 42s., 7s., 3s. 6d. 23. $\frac{5}{12}$. 24. 9 miles an hour.
25. 3d. for 14 lbs., 2d. for every extra 7 lbs. 26. $3\frac{9}{17}$ minutes.
27. 78. 28. 10, 7, 5 miles an hour, 70 miles. 29. 7 ft., 18 stone.
30. 7·2 cwt., 11·25 miles. 31. 40 yds., $60\frac{1}{2}$ yds. 32. 7, 5.
33. 9, 4 yards. 34. 32 yds. long, 27 yds. wide.
35. 88 in., 80 in. 36. 10 hours, 15 hours.
37. $20\frac{1}{2}$ ft., 16 ft. 38. 3 miles an hour. 39. $14\frac{1}{7}$.
40. 10 minutes, 15 minutes. 41. 3, 4, 5 miles an hour.
42. $15\frac{3}{4}$ oz., $16\frac{1}{4}$ oz. 43. 6 miles, 8 miles an hour. 44. £5. 14s.
45. $5\frac{1}{2}$, $6\frac{3}{5}$ hours. 46. 12 miles, 3 miles an hour, 4 miles an hour.
47. 8 miles, 16 miles, $4\frac{1}{2}$ miles an hour, $7\frac{1}{2}$ miles an hour.
48. $\frac{9}{19}$. 49. $1\frac{1}{2}$, $1\frac{1}{3}$, $1\frac{1}{4}$ minutes. 50. 10 gallons.

XXX. a. (p. 243).

1. $\cdot 5a^3b$. 2. $\dfrac{\cdot 01x^3}{y}$. 3. $\cdot 5x^2y$. 4. $\dfrac{x^5}{\cdot 08}$.
5. $2(a-b)$. 6. $\dfrac{1}{x-3}$. 7. $2x\pm 3y$. 8. $1\pm 2a^2b$.
9. $x\pm\dfrac{1}{x}$. 10. $x\pm\dfrac{5a}{4}$. 11. $1\pm(a-b)$. 12. $\dfrac{a}{b}$.
13. x. 14. $2a$. 15. $2x^2\pm\dfrac{1}{2x^2}$. 16. $2x^2\pm\dfrac{1}{x^2}$.
17. 4, 5. 18. -3, 1. 19. 5, 2. 20. 4, -5.
21. 0, -5. 22. $\pm\frac{4}{5}$. 23. $-\frac{1}{2}$, $\frac{1}{2}$. 24. $1\frac{1}{4}$, $2\frac{1}{3}$.
25. $1\frac{1}{7}$, $-\frac{1}{9}$. 26. a, -3. 27. 1. 28. $1\frac{2}{5}$, $4\frac{2}{3}$.
29. 4, -2. 30. -1. 31. 1, -2. 32. 1.
33. $1\frac{1}{2}$. 34. $\frac{1}{2}$. 35. 2. 36. $\frac{1}{2}$. 37. $\frac{1}{5}$.

XXX. b. (p. 244).

1. $\dfrac{2ax}{4x^2-9a^2}$, 0. 2. $a+b-1$, $a^2+b^2+c^2+2ab-2ac-2bc$, $a^3+3a^2b+3ab^2+b^3$.
3. $\pm\frac{1}{3}$. 4. 2·83, 3·61. 5. $x=3$, $-2\frac{1}{3}$. $y=1$, $-1\frac{2}{7}$. 7. $\frac{7}{18}$.

XXX. c. (p. 244).

1. $\dfrac{2x}{(x-a)(x-b)(x+b)}$. 2. ± 10. 3. 3, -2. 4. $5x^2-7x+4$.
5. $x=2\cdot5$, $y=6\cdot25$. 6. $x=3$, 4, $y=4$, 3. 7. 7062.

XXX. d. (p. 245).

1. $\dfrac{a^2}{(a+2b)(a-3b)}$. 2. x^2-x-4, $a^2+4b^2+c^2-4ab+2ac-4bc$, $a^3+6a^2b+12ab^2+8b^3$.
3. 1, 2 are the roots. 4. $7\cdot40$, $7\cdot65$. 5. $x=3$, -8, $y=4$, $-1\frac{1}{2}$.
6. 1, 4. 7. Half a minute.

XXX. e. (p. 245).

1. $\dfrac{5}{(x-1)(x+2)(x+3)}$. 2. ± 12. 3. $1\frac{1}{2}$, $-1\frac{1}{3}$. 4. $x<2\frac{1}{2}>-3\frac{1}{2}$.
5. $x=\pm 1$, $y=\pm 2$. 6. $4x^2-2x+\dfrac{1}{x}$. 7. £15. 15*s*.

XXX. f. (p. 245).

1. 1. 3. $-2\cdot8$, $2\cdot3$. 4. $12\cdot25$, $-6\cdot25$. 5. $x=8$, $y=1$.
7. 2340.

XXX. g. (p. 246).

1. x^2. 3. $2\cdot15$, $-1\cdot4$. 4. $2\cdot83$. 5. $x=6$, $\frac{1}{2}$, $y=\frac{1}{2}$, $-2\frac{1}{4}$.
6. x^2-6x+1. 7. $3\frac{2}{7}$ miles an hour.

XXX. h. (p. 246).

1. $(x^2+3x+3)(x^2-3x+3)$, $(8x-1)^3(1-a)(1+a+a^2)$.
2. $\dfrac{x(a-c)}{(x+a)(x+c)}$. 3. $(x^2-y^2)^2+(x^2-y^2)z^2+z^4$. 4. 68*s*., 86*s*., 98*s*.
5. $2\cdot6$, $-1\cdot6$. 6. $x=\pm 1$, $y=\pm 2$. 7. 80, £32.

XXX. k. (p. 246).

1. $16a^6-36a^4b^2-108a^3b^3-162a^2b^4+486ab^5+729b^6$.
2. $\dfrac{(c+a-b)(c-a+b)(c+a-b)(c-a+b)}{4a^2b^2}$. 3. 6. 4. $2\cdot56$, $-1\cdot56$.
5. 2.54 p.m., 1.51 p.m., 3.57 p.m. 6. $x=\pm 4$, ± 1. $y=\pm 1$, ± 4. 7. Friday.

XXX. l. (p. 247).

1. $6x^5-x^4+10x^3-14x^2-25$. 3. 5, 3. 4. $x=6\cdot37$, $\cdot63$, $y=\cdot63$, $6\cdot37$.
5. They meet in 4 hours, 42 miles from home. They are 10 miles apart in 5 hours.
6. $x=\pm 5$, $\pm 2\sqrt{3}(=\pm 3\cdot46)$, $y=\pm 4$, $\mp\sqrt{3}(=\mp 1\cdot73)$. 7. $\dfrac{p}{q+r}$ hours.

XXX. m. (p. 247).

1. $\frac{1}{(a-b)^2}$. 2. 3. 3. $\frac{x^2-3x+2a}{x-3}$.

4. £19. 18s., £41, £57. 8s. 5. $x=6, -2,$ $y=6, 2.$

6. $x=\pm 2, y=\pm 1.$ 7. 39 ft. long, 31 ft. wide.

XXX. n. (p. 248).

1. $\frac{a-b}{a+b}$. 2. $4, \frac{1}{4}$. 3. $2a^2b(a+b), (x-2)(x-3)(x-5)$.

4. $x=0, 4,$ $y=0, -8.$ 5. $12, -1\cdot 5.$ 6. $x=\pm\frac{1}{2}, \pm 9\frac{1}{2},$ $y=\mp 2, \pm 7.$ 7. One mile.

XXX. p. (p. 248).

1. $bx+ay+1$. 2. $-8, -12$.

3. $(a-b)(a+b-c)(a+b+c), (x^2-xy-y^2)(x^2+xy-y^2)$.

4. $4\cdot 54, -1\cdot 54$. 5. $25\frac{3}{4}$ miles an hour.

6. $x=\frac{1}{3}, -\frac{1}{8}.$ $y=\frac{1}{4}, -\frac{1}{7}.$ $z=\frac{1}{6}, \frac{1}{28}.$ 7. 480 apples, 400 pears.

XXXI. a. (p. 249).

1. $-a-b$. 2. $\frac{1}{ab}$. 3. $\frac{ac}{b}$.

4. $\frac{2(a-2b)(2a-b)}{a+b}$. 5. $\frac{2}{a^2b^2+b^2c^2+c^2a^2}$. 6. $\frac{a+c}{2}$.

7. $a+b$. 8. $\frac{a+c}{b}$. 9. $\frac{a^2+b^2}{a+b}$.

10. abc. 11. $\frac{2ab}{a+b}$. 12. $\frac{pr}{q}$.

13. 0. 14. $\frac{a^2+b^2}{a+b}$. 15. $\frac{ab}{a+b}$.

16. $\frac{a^6+1}{a(a^2-a^4-2)}$. 17. $-\frac{ab}{a^2-ab+b^2}$. 18. $\frac{ad+bc-2bd}{a-b+c-d}$.

19. $\frac{2ab}{a-b}$. 20. $-\frac{a+b}{2}$. 21. $a+3b$.

22. $-\frac{a+c}{2}$. 23. $\frac{a+b}{2}$. 24. $\frac{2ab}{a+b}$. 25. $\frac{a+b}{2}$.

XXXI. b. (p. 251).

1. $x=a+1, y=a-1$. 2. $x=c, y=-a$.

3. $x=3a-b, y=a+3b$. 4. $x=\frac{s+t}{2a}, y=\frac{s-t}{2b}$.

5. $x=\frac{a^2+ab+b^2}{a+b}, y=\frac{ab}{a+b}$. 6. $x=a+b, y=a-b$.

7. $x=c, y=-a$.

8. $x=\frac{a-c}{a-b}, y=\frac{a-c}{b-c}$.

9. $x=\frac{a+b}{a-b}, y=\frac{a-b}{a+b}$.

10. $x=\frac{3b}{2}, y=-\frac{a}{2}$.

11. $x=\frac{b+c-a}{a+b-c}, y=\frac{a+c-b}{a+b-c}$.

12. $x=\frac{c(a^2+b^2)}{a^2-b^2}, y=\frac{c(a^2+b^2)}{2ab}, x=0, y=0$.

13. $x=\frac{c-a}{c+a}, y=\frac{a-c}{2(c+a)}$.

14. $x=\frac{a+b+c}{a+b}, y=\frac{a+b}{c}$.

15. $x=a, y=b$.

16. $x=\frac{a^2-b^2}{ap-bq}, y=\frac{a^2-b^2}{aq-bp}$.

17. $x=a+b, y=a-b$.

18. $x=\frac{bc-d}{ab-1}, y=\frac{ad-c}{ab-1}$.

19. $x=\frac{a^2-bc}{a}, y=\frac{b^2-ac}{b}$.

20. $x=6a+b, y=2a-b$.

21. $x=\frac{a}{a^2+1}, y=\frac{-1}{a^2+1}$.

22. $x=\frac{b+c-a}{2a}, y=\frac{c+a-b}{2b}, z=\frac{a+b-c}{2c}$.

23. $x=\frac{\pm a}{\sqrt{la^2+mb^2+nc^2}}, y=\frac{\pm b}{\sqrt{la^2+mb^2+nc^2}}, z=\frac{\pm c}{\sqrt{la^2+mb^2+nc^2}}$.

24. $x=\frac{2abc}{ab-bc+ac}, y=\frac{2abc}{ab+bc-ac}, z=\frac{2abc}{bc+ac-ab}$.

XXXI. c. (p. 252).

1. $x=5a, -3a$.
2. $x=2a, 3a$.
3. $x=\frac{1}{a}, \frac{c}{b}$.
4. $x=a, b$.
5. $x=a\pm\frac{1}{a}$.
6. $x=\frac{1}{a}, -\frac{q}{p}$.
7. $x=\pm a$.
8. $x=\frac{b}{a}$.
9. $x=\frac{1}{a}, \frac{1}{b}$.
10. $x=-\frac{1}{a}, \frac{1}{b}$.
11. $x=4b, -3b$.
12. $x=\frac{f^2}{ag}$.
13. $x=\frac{5a}{2}, \frac{3a}{10}$.
14. $x=3a, \frac{3a}{2}$.
15. $x=-2a, 2a+2b$.
16. $x=\frac{a-b}{2}, \frac{a+b}{2}$.
17. $x=\frac{a+b}{a-b}, \frac{a-b}{a+b}$.
18. $x=a, b$.
19. $x=a+1, \frac{2}{a-1}$.
20. $x=\frac{1}{3}[a+b+c\pm\sqrt{a^2+b^2+c^2-bc-ac-ab}]$.
21. $x=\frac{1}{2}\left(a\pm\frac{1}{b}\right)$.
22. $x=a+b, \frac{a+b}{2}$.
23. $x=0, a+b$.
24. $x=1, \frac{-2ab}{a^2+2ab-b^2}$.
25. $x=b, 2a-b. \quad y=a, 2b-a$.

XXXI. d. (p. 254).

1. 11.
2. 2.
3. 7.
4. $1\frac{1}{12}$.
5. 1.
6. ±5.
7. $0, \frac{4}{3}$.
8. 3.

9. $\frac{1}{2\sqrt{10}}$. **10.** 8. **11.** -5. **12.** -4.

13. 4. **14.** 8. **15.** $\frac{(a^2+b^2)^2}{(a+b)^2}$. **16.** ± 5.

17. $a+2b$. **18.** $-\frac{3}{28}$. **19.** $\frac{11}{7}$ **20.** $\frac{b}{a}$.

21. 16. **22.** 0. **23.** $\frac{2}{3}$. **24.** $\frac{a^2}{16}$.

25. a^2+b^2. **26.** $1, -4$. **27.** $0, 5$. **28.** -1.

29. $2, -4$. **30.** $2, -\frac{4}{3}$. **31.** $\frac{1}{2}(3\pm\sqrt{5})$. **32.** $\frac{15}{2}, -1$. **33.** $2, -5$.

XXXI. e. (p. 256).

1. $x=6, 4,$ $y=4, 6,$ $z=5, 5.$

2. $x=9, 1,$ $y=3, 3,$ $z=1, 9.$

3. $x=-\frac{1}{2}, \frac{1\pm\sqrt{29}}{4}.$

4. $x=\pm\frac{3\sqrt[4]{2}}{2}, \pm 3,$ $y=\pm\sqrt[4]{2}, \pm 1.$

5. $x=\pm 4,$ $y=\pm 2.$

6. $x=a, \frac{a+b}{2},$ $y=b, \frac{a+b}{2}.$

7. $x=0, \pm\frac{2c}{\sqrt{3}},$ $y=\pm c, \mp\frac{c}{\sqrt{3}}.$

8. $x=1, 1, 2, 2, 4, 4,$ $y=2, 4, 1, 4, 1, 2,$ $z=4, 2, 4, 1, 2, 1.$

9. $x=\pm\sqrt{\frac{ab^2}{2b-a}}.$

10. $x=\frac{ab(c+d)-cd(a+b)}{ab-cd}.$

11. $x=\pm\left(\frac{1}{b}+\frac{1}{a}\right), \pm\left(\frac{1}{b}-\frac{1}{a}\right),$ $y=+\left(\frac{1}{b}-\frac{1}{a}\right), \pm\left(\frac{1}{b}+\frac{1}{a}\right).$

12. $x=\pm\sqrt{6},$ $y=\pm\frac{\sqrt{6}}{2},$ $z=\pm\frac{\sqrt{6}}{3}.$

13. $x=\mp\frac{23}{12},$ $y=\pm\frac{31}{12},$ $z=\pm\frac{41}{12}.$

14. $x=\pm 3, y=\pm 1.$

XXXI. f. (p. 259).

1. (0, 7)(5, 5)(10, 3)(15, 1). **2.** (0, 5)(3, 3)(6, 1).

3. (5, 1)(3, 6)(1, 11). **4.** (7, 8)(10, 1)(4, 15)(1, 22). **5.** (2, 3).

6. (11, 10)(24, 3). **7.** 7. **8.** 8. **9.** 6. **10.** 6.

12. (1, 7)(3, 4)(5, 1). **13.** (1, 13)(2, 8)(3, 3)(0, 18).

14. (0, 12)(4, 9)(8, 6)(12, 3)(16, 0). **15.** (1, 3)(8, 1).

16. (0, 10)(3, 8)(6, 6)(9, 4)(12, 2)(15, 0).

17. (2, -5)(4, -4)(6, -3)(8, -2)(10, -1). **18.** (3, -6)(6, -4)(9, -2).

19. (1, -3)(2, -2)(3, -1). **20.** (-3, -6)(-6, -4)(-9, -2).

21. (-3, -4). **22.** (-2, -10)(-4, -8)(-6, -6)(-8, -4)(-10, -2).

23. 2 at 5*s*. each, 4 at 7*s*. **24.** 6 geese, 4 turkeys.
25. 30 ways. **26.** 27, 32.
27. Give 10 four-shilling pieces, receive 2 half-crowns.
28. 4 ways. **29.** $(\frac{1}{4};\ \frac{13}{7})$, $(\frac{5}{4};\ \frac{6}{7})$. **30.** $x=13p+7$, $y=9p$.
31. 3 ways. **32.** 35, 4. **33.** 3 ways.

XXXII. (p. 266).

1. $x^2-7x+10=0$. **2.** $x^2+x-20=0$. **3.** $4x^2-1=0$.
4. $x^2+3x=0$. **5.** $x^2+ax-6a^2=0$. **6.** $x^2-2ax+a^2-1=0$.
7. $a^2x^2-2a^2x+a^2-1=0$. **8.** $x^2-2mx+n=0$. **9.** $lx^2+mx+n=0$.
10. $x^2-6x+6=0$. **11.** $25x^2-40x+13=0$. **12.** -25.
13. p^2-4q must be a perfect square. **15.** $\frac{p\pm\sqrt{p^2-4q}}{2}$, p, q.
16. (i) $\pm\frac{\sqrt{b^2-4ac}}{a}$. (ii) $\frac{b^2-2ac}{a^2}$. (iii) $\frac{b(3ac-b^2)}{a^3}$. (iv) $\frac{(b^2-2ac)^2}{a^4}-\frac{2c^2}{a^2}$.
17. $x^2-2px+4q=0$. **18.** $ax^2-bx+c=0$.
19. $ax^2+3bx+9c=0$. **20.** $acx^2-(b^2-2ac)x+ac=0$.
21. $acx^2-2(b^2-2ac)x+4ac=0$. **22.** $a^2x^2+abx+9ac-2b^2=0$.
23. $a^2cx^2-b(3ac-b^2)x+ac^2=0$. **24.** $a=-\frac{2}{3}$.
26. $x^2+4px-p^2=0$. **27.** $a^2(x^2+1)+(b^2+2a^2)x=0$.
28. $p(3q-p^2)$. **29.** $p^2x^2+px(q-r)-qr=0$. $\frac{(q^2-2pr)^2}{p^4}-\frac{2r^2}{p^2}$.
30. $a^2x^2-(b^2-2ac)x+c^2=0$. **31.** $x^2-(p^2+2q)x+q^2=0$.
32. $k=-2$. **34.** $(p'-p)(pq'-p'q)=(q-q')^2$.
39. $a^3x^2+2b(4b^2+3ac)x-c^3=0$. **40.** $\frac{b}{c}=\frac{3}{2}\frac{3}{7}$. **43.** (i) ac. (ii) c^2.
44. $\frac{b^2-2ac}{a^2c^2}$. **49.** $5a$. **50.** $\frac{1}{3}$.

XXXIII. a. (p. 268).

1. $(2x-5y)(3x-4y)$, $(x^2-3xy+y^2)(x^2+3xy+y^2)$,
$(x-1)(x+1)(x^2+x+1)(x^2-x+1)$.
2. 0. **3.** $8z(2z-1)$. **4.** $x^8(x^8-y^8)$.
5. (i) $2\frac{1}{3}$, $-\frac{2}{9}$. (ii) $x=\pm2$, ±1,
$y=\pm1$, ∓2.
6. 4 hrs. 35 min., 3 hrs. 48 min., 19·9 miles.
7. $x=-3$, $y=1\frac{1}{2}$, $z=4$. **8.** $x^2+3px+2p^2+q=0$.

XXXIII. b. (p. 268).

1. $(x+7)(x+9)$, $(y-a)(y+7a)(y-6a)$,
$x(x-1)(x+1)(x-2)(x+2)(x-3)(x+3)$.
2. $3x^2-7x-2$. **3.** 4. **4.** (i) $\frac{1}{ab}$. (ii) 1, 13.
5. £600. **6.** 90, 81, 71, 62, 41, 21.

XXXIII. c. (p. 269).

1. $367a-114b+690c$, 1082. 2. 0. 3. x^2-7x+2. 4. $-\frac{1}{2}$, 3.

5. (i) $-\frac{ac}{b}$. (ii) $x=2, 1\frac{1}{2}$, $y=1, 1\frac{1}{2}$. 6. £30. 7. $-1, -\frac{1}{2}$.

XXXIII. d. (p. 269).

1. $x^3-3x^2+11x-8$. 2. $\frac{a}{b}-1-\frac{b}{a}$. 3. $2x^2+3x-5$.

4. $20x$ yds., $\frac{15x}{22}$ miles, $\frac{15xy}{22}$ miles, $\frac{22y}{15x}$ hours.

5. (i) $x=0, 7, -2\frac{1}{21}$. (ii) $x=\frac{1}{3}, y=\frac{1}{2}$.

6. In $37\frac{1}{2}$ secs. 7. $x=1\cdot5$, max. value $2\cdot25$.

XXXIII. e. (p. 270).

2. x^6-y^6. 3. n^2+3n+1. 4. $-\frac{(x+y-z)^2}{2yz}$.

6. (i) $a-b$. (ii) 2·63, 1·37. 7. 15, 12 miles per hour.

8. $\frac{2}{3}, \frac{1}{3}$.

XXXIII. f. (p. 270).

1. x^2+3y^2. 2. $x(x-4)(4x-7)$, $(y+3)(y-3)(y^2+20)$, $(a^2+3b^2)(a^2-3ab+3b^2)(a^2+3ab+3b^2)$.

3. (i) $\frac{ab}{b-a}$. (ii) $x=\pm 4$, $y=\pm 3$. 4. $4a$, 4β.

6. 48 minutes. 7. $-(a+b+c)$.

XXXIII. g. (p. 271).

1. $2x^3+3x^2+8x+25$, remainder 74. 2. ·618.

3. 14/8, 14/-. 4. (i) $-4(a^2+b^2)$. (ii) 0.

5. (i) $\frac{2ab}{a+b}$. (ii) $x=\frac{ac}{a+b}$, $y=\frac{bc}{a+b}$. 6. £26, £50, £64.

7. $ax^2-2bx+4c=0$.

XXXIII. h. (p. 271).

1. $x^m(a+bx^2)$. 2. -30.

3. (i) a^2-ab+b^2. (ii) $(a^2+ab+b^2)(a^2-ab+b^2)(a^2+ab-b^2)$.

4. $x=2$, $y=5$ are common roots. 5. $x^2+3px-9q=0$.

6. (i) 2, $-3\frac{1}{2}$. (ii) $x=\frac{2}{3}, -\frac{1}{2}$, $y=-\frac{1}{2}, \frac{2}{3}$. 7. 41, 28 miles per hour.

XXXIII. k. (p. 272).

2. $\cdot 31, -\cdot 81$. 3. $\dfrac{1}{b-c}$. 4. $\frac{20}{21}$.

5. 5/17/-, 6/6/-, 7/12/-. 6. $x = \pm(a \pm b)$, $y = \pm(a \mp b)$.

7. $ax^2 + (b - 2am)x + am^2 - bm + c = 0$.

XXXIII. l. (p. 272).

1. $(a^2 - 12b)(a^2 + 4b)$, $(a + c)(ac + b^2)$. 2. $a^6 - 64b^6$.

4. $161\frac{1}{2}$. 5. $x = 1$, $y = 2$, $z = 3$.

6. $b^2 < ac$. 7. 43, 18 miles per hour.

XXXIII. m. (p. 272).

1. $(b - c)$. 2. $(2x + 7)(9x - 5)$, $(a - c)(a + c - 2b)$, $(x - b)(x - 3b)(x - 5b)$.

3. $x^4 + 7x^3 + 2x - 3$. 4. 3·61.

5. (a) $x = \frac{1}{3}, \frac{2}{3}$, $y = \frac{2}{3}, \frac{1}{3}$. (b) $\dfrac{a}{b}, \dfrac{b}{a}$. 7. 25, 44, 64.

XXXIII. n. (p. 273).

1. $(ac - bd)^2 + (ad - bc)^2 = (ac - bd)(ad - bc)$. 2. (i) 0, (ii) $\dfrac{n^2(3n^2+1)}{4}$.

3. $(3x + 2)(x - 2)(2x - 1)(2x + 1)$. 4. 3·5.

5. £800. 6. (i) $-\dfrac{bc}{a}$. (ii) $x = \pm\sqrt{\dfrac{a}{2b}}$, $y = \pm\sqrt{\dfrac{b}{2a}}$.

7. $acx^2 + (ab + 2ac - b^2)x + a(a - b + c) = 0$.

XXXIII. p. (p. 273).

1. $x^2(x^2 - 1)(x^4 + x^2 + 1)$. 2. $x^2 + y^2 + z^2 + yz - zx + xy$.

3. $8x^6 + 6x^5 - 4x^4 - 37x^3 - 15x^2 + 7x + 35$. 4. $-3\cdot 83$, $1\cdot 83$.

5. (i) 0, $\frac{4}{5}$. (ii) $x = -\frac{3}{4}, 1\frac{3}{4}, -1\frac{1}{4}, \frac{1}{4}$, $y = -\frac{3}{4}, 1\frac{3}{4}, \frac{1}{4}, -1\frac{1}{4}$. 6. 5.

XXXIII. q. (p. 274).

2. (i) $x = 0$, $\dfrac{ad - bc}{a - c}$, $y = 0$, $\dfrac{bc - ad}{b - d}$. (ii) $\dfrac{a - b}{2}$. 4. $(64x^6 - 729)(3x + 2)$.

5. 9·75. 6. 35/-.

XXXIII. r. (p. 274).

1. $(x-1)(x+1)$, $(x-7)(x+1)$, $x(x-1)(x-2)$, $(3x-1)(x-2)$,
L.C.M. $x(x-1)(x+1)(x-7)(x-2)(3x-1)$.

2. (i) 3. (ii) $a+b$. 3. $5{\cdot}53$, $-2{\cdot}53$. 4. $2a^2-3ab+2b^2=0$.

5. A was elected by a majority of 5. 6. $x=\pm\sqrt{2}(\pm 1{\cdot}41)$, ± 5,
$y=\mp 4\sqrt{2}(5{\cdot}66)$, ± 3.

7. $\dfrac{a^3-3ab+2c}{6}$.

XXXIII. s. (p. 275).

1. $2(x^2+y^2+z^2-xy-yz-xz)$. 2. 1, $-\dfrac{a+2b}{2a+b}$.

3. (i) n^2. (ii) $n^2+(n-1)^2$.

4. $x^2-(m+n)(p^2-2q)x+q^2(m^2+n^2)+mn(p^2-4p^2q+2q^2)=0$.

5. $x<-3\frac{1}{2}$ or $>2\frac{1}{2}$.

6. $x=1, 1, 2, -2, 2, -2$,
$y=2, -2, 1, 1, -2, 2$,
$z=-2, 2, -2, 2, 1, 1$.

7. 14, 8 miles per hour.

XXXVI. a. (p. 298).

1. $\sqrt{18}$. **2.** $\sqrt{50}$. **3.** $\sqrt{45}$. **4.** $\sqrt{75}$. **5.** $\sqrt{294}$.
6. $\sqrt{32}$. **7.** $\sqrt{18}$. **8.** $\sqrt{20}$. **9.** $\sqrt{7}$. **10.** $\sqrt{27}$.
11. $\sqrt{32}$. **12.** $\sqrt{54}$. **13.** $2\sqrt{3}$. **14.** $2\sqrt{2}$. **15.** $4\sqrt{2}$.
16. $5\sqrt{3}$. **17.** $7\sqrt{5}$. **18.** $9\sqrt{3}$. **19.** $3\sqrt[3]{3}$. **20.** $-3\sqrt[3]{3}$.
21. $2\sqrt[3]{2}$. **22.** $2\sqrt[3]{4}$. **23.** $10\sqrt{5}$. **24.** $13\sqrt{3}$. **25.** $2\sqrt{2}$.
26. $3\sqrt{3}$. **27.** $4\sqrt{2}$. **28.** $7\sqrt{3}$. **29.** $4\sqrt{5}$. **30.** $\sqrt{2}$.

XXXVI. b. (p. 300).

1. $41\sqrt{3}$. **2.** $7\sqrt{5}+17\sqrt{2}$. **3.** $-4\sqrt{6}$. **4.** $-7\sqrt{5}$.
5. $5\sqrt[3]{3}$. **6.** $28\sqrt[3]{4}$. **7.** $8\sqrt{3}$. **8.** $3\sqrt{6}+1$.
9. 2. **10.** 12. **11.** $1\frac{1}{2}$. **12.** 3.
13. $5+2\sqrt{6}$. **14.** $4\frac{1}{2}$. **15.** 10. **16.** 8.
17. $17+\sqrt{3}$. **18.** $36-13\sqrt{6}$. **19.** $29-2\sqrt{6}$.
20. $42+\sqrt{105}-6\sqrt{21}-3\sqrt{5}$. **21.** 87. **22.** 33.
23. $\frac{\sqrt{30}}{3}$. **24.** $\frac{\sqrt{35}}{2}$. **25.** $\sqrt{5}$. **26.** $\frac{5\sqrt{2}}{2}$.
27. $\sqrt{2}-1$. **28.** $\frac{2+\sqrt{2}}{2}$. **29.** $3-2\sqrt{2}$. **30.** $\frac{a+b-2\sqrt{ab}}{a-b}$.
31. $3-2\sqrt{2}$. **32.** $\sqrt{5}-1$. **33.** $\sqrt{5}-\sqrt{2}$. **34.** 2.
35. $\frac{27+11\sqrt{6}}{6}$. **36.** $\frac{a-\sqrt{a^2-b^2}}{b}$. **37.** $\frac{\sqrt{5}+1}{4}=0{\cdot}81$.
38. $12+8\sqrt{2}=23{\cdot}31$. **39.** $\frac{107+42\sqrt{2}}{89}=1{\cdot}87$. **40.** $\frac{7\sqrt{14}-13}{11}=1{\cdot}20$.
41. $3-\frac{\sqrt{6}}{3}=2{\cdot}18$. **42.** $5-2\sqrt{6}=0{\cdot}10$. **43.** $\sqrt{6}+\sqrt{3}-\sqrt{2}-1=1{\cdot}77$.
44. $\frac{3}{4}(\sqrt{7}+\sqrt{3})=3{\cdot}28$. **45.** $\frac{9\sqrt{6}+3\sqrt{3}-3\sqrt{2}-1}{17}=1{\cdot}29$.
46. $5\sqrt{3}+3\sqrt{2}=12{\cdot}90$. **47.** $\sqrt{5}+1=3{\cdot}24$.
48. $\frac{3\sqrt{2}+2\sqrt{3}+\sqrt{30}}{12}=1{\cdot}10$. **49.** $3+2\sqrt{3}+\sqrt{21}=11{\cdot}05$.
50. $\frac{5\sqrt{3}+3\sqrt{5}-2\sqrt{30}}{30}=0{\cdot}15$. **51.** $\frac{2+\sqrt{6}+\sqrt{2}}{4}=1{\cdot}47$.

XXXVI. c. (p. 302).

1. $\sqrt{3}+1=2{\cdot}73$. **2.** $\sqrt{6}+1=3{\cdot}45$. **3.** $3-\sqrt{3}=1{\cdot}27$.
4. $3+\sqrt{2}=4{\cdot}41$. **5.** $2\sqrt{7}+\sqrt{2}=6{\cdot}71$. **6.** $3-2\sqrt{2}=0{\cdot}17$.
7. $\sqrt{7}+\sqrt{5}=4{\cdot}88$. **8.** $2\sqrt{5}-2\sqrt{3}=1{\cdot}01$. **9.** $\frac{\sqrt{5}}{2}-1=0{\cdot}12$.

10. $2\sqrt{5}+\sqrt{7}-7\cdot 12$ 11. $2\sqrt{13}-7=0\cdot 21$. 12. $\sqrt{2}+\frac{1}{6}=1\cdot 58$.
13. $\sqrt[4]{2}(\sqrt{3}-1)=\cdot 87$. 14. $3\sqrt{3}-\sqrt{6}=2\cdot 75$. 15. $\sqrt{3}+\sqrt{2}$.
16. $\sqrt{2}+1$. 17. $\sqrt{5}-1$. 18. $\sqrt{7}-\sqrt{5}$.
19. $0\cdot 3090$. 20. 5. 21. $2\sqrt{5}=4\cdot 472$.
22. $\frac{1}{3}\sqrt{6}=0\cdot 816$ 23 1. 24. $\sqrt{2}+1$.
25. $\frac{\sqrt{3}}{30}(7\sqrt{5}-11\sqrt{2})$; $\frac{\sqrt{3}}{18}(9\sqrt{5}-11\sqrt{2})$; $0\cdot 117$. 26. 50.
27. 1 or $\frac{c^2}{b^2}$. 28. $1+\sqrt{2}+\sqrt{3}$. 29. $\sqrt{2(1+a)}$. 30. 1.

XXXVII. a. (p. 307).

1. 4. 2. $\frac{1}{8}$. 3. 4. 4. 4. 5. 3. 6. $\frac{1}{2}$.
7. 1. 8. $a^{\frac{3}{2}}$. 9. $a^{\frac{5}{2}}$. 10. $a^{\frac{1}{2}}$. 11. $2x$. 12. $\frac{1}{a}$.
13. 4. 14. 9. 15. 8. 16. $\frac{1}{5}$. 17. 2. 18. 36.
19. $\frac{1}{4}$. 20. 81. 21. $\frac{1}{2}$. 22. 2. 23 64. 24. $\frac{1}{64}$.
25. 27. 26. 3. 27. 4. 28. $\frac{1}{2}$. 29. 25. 30. $\frac{1}{25}$.
31. 1. 32. 5. 33. $a+2\sqrt{ab}+b$. 34. $a-b$.
35. x^2+2+x^{-2}. 36 x^{2a}. 37. $x^{a^2-b^2}$.
38 $e^{2x}+2+e^{-2x}$. 39. $x^{\frac{1}{2}}-2x^{\frac{1}{4}}y^{\frac{1}{4}}+y^{\frac{1}{2}}$. 40. $4x^2-2+\frac{x^{-2}}{4}$.

XXXVII. b. (p. 308).

1. $a^{\frac{1}{3}}$. 2. $a^{\frac{1}{5}}b^{\frac{2}{5}}$. 3. $3b^{\frac{1}{2}}c^{-\frac{1}{3}}$. 4. $a^{\frac{10}{3}}$; $x^{-\frac{1}{2}}y^{-\frac{1}{4}}$; $2^{\frac{1}{2}}a^{-\frac{3}{2}}$.
5. 4; $\frac{1}{5}$. 6. $\frac{1}{9}$; 343. 7. $\frac{bc}{a}+\frac{c}{a^2b}+\frac{a}{bc}$. 8. 8
9. $\frac{1}{8}$. 10. $\frac{1}{17}$. 11. $\frac{1}{4}$. 12 $\frac{1}{81}$. 13. 125.
14. 200. 15. 49. 16. $\frac{x^n}{a}$. 17. $\frac{a^3}{2b^2}$. 18. $\frac{3c}{b^2}$.
19. $\frac{x^6}{64y^4}$. 20. $\frac{2b^2}{a}$. 21. $\frac{ab}{b-a}$. 22. $a-\sqrt{ab}+b$.
23. $\frac{1}{16}$ 24. 256. 25. $\frac{2}{5}$. 26. $10y^{\frac{3}{2}}+29x^{\frac{1}{4}}y+16x^{\frac{1}{2}}y^{\frac{1}{2}}-7x^{\frac{3}{4}}$.
27. x^3+1+x^{-3} 28. $a^{-2}+a^{-1}b^{-1}+b^{-2}$. 29. a^3-b^2.
30. $a^2+4ab^{\frac{1}{4}}+4b^{\frac{1}{2}}-9c$. 31. $x-y^4$.
32. $a^4-9a^2b^{-2}-24ab^{-1}c^{-2}-16c^{-4}$. 33. $4x^2+6xy^{-1}+9y^{-2}$.
34. $x^{-\frac{5}{2}}-2x^{-2}y^{\frac{1}{3}}+4x^{-\frac{3}{2}}y^{\frac{2}{3}}-8x^{-1}y+16x^{-\frac{1}{2}}y^{\frac{4}{3}}-32y^{\frac{5}{3}}$. 35. $(a^{\frac{1}{4}}+b^{\frac{1}{4}})^2$.
36. $\sqrt[3]{2}$. 37. $a^{\frac{5}{2}}+a^2b^{-\frac{1}{2}}+a^{\frac{3}{2}}b^{-1}+ab^{-\frac{3}{2}}+a^{\frac{1}{2}}b^{-2}+b^{-\frac{5}{2}}$.
38. $a^{\frac{2}{3}}+a^{\frac{1}{3}}b^{\frac{1}{3}}+b^{\frac{2}{3}}$. 39. $\frac{1}{4}(a^{\frac{1}{2}}+a^{-\frac{1}{2}})(5a-4+5a^{-1})$. 40. $a^{\frac{1}{4}}-b^{\frac{1}{2}}$.

XXXVIII. a. (p. 316).

2. 2 : 3. **3.** 13 : 23. **4.** 2·54. **5.** ·7. **6.** 4 : 7. **7.** 5 : 8.
8. 4. **9.** 3 : 1 or 1 : 3. **10.** 10. **14.** 21, 28.
15. 25, 20. **16.** 40, 45. **17.** 32, 60. **18.** 31·25, 33·75.
19. 2 : 3. **20.** 60. **21.** $3\frac{1}{7}$. **23.** 5. **24.** 10.
25. $\frac{ad-bc}{c-d}$. **27.** $\frac{43}{59}$. **29.** $\frac{3}{5}$ greatest, $\frac{6}{11}$ least. **30.** ·6, ·56, ·59.
34. $\frac{x+y}{y}$. **35.** 80·1 feet. **37.** £200, £150.
38. 1 inch represents 3 feet ; 1 : 1296. **44.** $\frac{1}{2}$. **45.** $\frac{205}{316}$. **46.** $\frac{7}{3}$
50. 5^2. **56.** 40, 16. **57.** $\frac{3}{4}$ or 1. **59.** 16 : 5. **60.** 37 : 39.
61. −5 : 4 ; 5 : 1. **62.** $a^3+b^3+c^3-3abc$. **63.** $abc+2fgh-af^2-bg^2-ch^2$
64. $(a+b)(d-c):(a-b)(d+c)$. **65.** Scale $\frac{1}{17280}$; $\frac{121}{90}$ sq. in
66. 5 : 2 or 2 : 1. **67.** 11 : 24. **68.** 136. **70.** $3\frac{1}{3}$ miles.
71. 82, 65, 57. **72.** 6. **73.** 2 gallons, 14 gallons.

XXXVIII. b. (p. 321).

1. $a:c=d:b$, or $a:d=c:b$. **2.** 21 ; $\frac{25x^5}{2}$. **3.** bc ; $12yz$; $\frac{b^3c^3}{a^2}$.
9. 4, 3. **10.** 5, 15, 45. **11.** $\frac{5+3\sqrt{3}}{2}$. **12.** $2(\sqrt{5}+\sqrt{2})$.
18. $\frac{a+b}{x+y}=\frac{x+y}{a-b}$. **19.** $a\frac{\sqrt{5}-1}{2}$. **21.** 341, 68, 45.
22. 3, 12. **23.** $a:c$. **25.** $\frac{2p}{a}$.

XXXVIII. c. (p. 327).

1. $5x=6y$; 7·5, 4·2. **2.** $\frac{2}{7}$. **3.** 101·25.
4. 24. **5.** $y=4x+3$; 17. **9.** $\frac{4}{9}$ lb.
10. $xy=\frac{12}{25}(x^2+y^2)$. **11.** $y=\frac{14}{4-5x}$. **15.** 36.
16. £200. **17.** 12 oz. **18.** £688. 10*s*. 8*d*. **19.** 20 miles an hour.
20. 7 boys. **21.** 16 times what it was.
22. Approximately 1·414 feet away from the light.
23. 175 men. **24.** 8 feet. **25.** 8 hours a day. **26.** £1925.
27. $10\frac{1}{2}$ lb. **28.** 5·4 inches. **29.** 3240 lb. **30.** ·72 : 25.
31. £1470. **32.** 78·5 sq. m. ; 2·45 m. **33.** 336 feet.
34. £14. 4*s*. 5*d*. **35.** 44·5 ; $\frac{88}{37}$. **36.** 3 miles ; $10\frac{2}{3}$ feet.
37. 1·074 secs. **38.** $4000\times\frac{81}{1681}$ miles = 193 miles nearly. **39.** £34.
41. Expense $=\left(12+\frac{3}{4}x\right)$£. **42.** 80 lb. per sq. in. **43.** 48 : 5.

XXXIX. a. (p. 335).

1. 3. 2. 5. 3. 2. 4. 4. 5. 6. 6. 8.
7. 0. 8. 1. 9. 4. 10. 2. 11. 2. 12. 3.
13. 3. 14. 3. 15. 3. 16. 5. 17. 3. 18. -1.
19. 4. 20. $\frac{1}{2}$. 21. $\frac{1}{3}$. 22. 5. 23. 5. 24. $\frac{1}{2}$.
25. $\frac{1}{3}$. 26. $\frac{4}{3}$. 27. $\frac{5}{2}$. 28. $\frac{1}{3}$. 29. -3. 30. $-\frac{1}{2}$.
31. $\log a + \log b + \log c$. 32. $2\log a + 3\log 5 + 5\log c$.
33. $\log a + 2\log b - \log c - 2\log d$. 34. $2\log 2$. 35. $\log 2 + \log 3$.
36. $3\log 2$. 37. $1 + \log 3 - \log 2$. 38. $\log 2 + 2\log 3$. 39. 1.
40. $\log 6$. 41. $2\log 2$. 42. $\log 3$. 43. $\log 6$. 44. $3 + \log 3$.
45. 6. 46. 3. 47. 5. 48. -1. 49. -2. 50. 2.
51. 3. 52. 1. 53. 1. 54. $\bar{2}$. 55. 3. 56. 2.
57. 3. 58. 1. 59. 0. 60. 4. 61. 5. 62. $-3 + x$.
63. 3. 64. 1. 65. 2. 68. $\log(a+b) + \log(a-b)$. 69. 2035.
70. 2035. 71. 516. 72. 1308. 73. 0·4136. 74. 5·4136.
75. 3·4136. 76. $3 - 3\log 2 + 4\log 3 + 2\log 7$. 77. 2. 78. 3.
79. 1. 80. 5. 81. 1·3010. 82. 3·3010. 83. 1·3010.
84. 4·3010. 85. 5·3010. 86. 3·3010. 87. 0·3736. 88. 2·3736.
89. 5·3736. 90. 1·3736. 91. 3·3736. 92. 5·3736.

XXXIX. b. (p 337).

1. 1·2068. 2. 3·9356. 3. 3·4042. 4. 2·4925. 5. $\bar{8}$·3855.
6. 5·3980. 7. 1·8474. 8. 1·5796. 9. 1·4615. 10. 13.
11. 7. 12. 8. 14. $\log 5$. 15. 0. 16. 26.
17. 2^{13}. 18. 2^{23}. 19. $2^{\cdot 9}$. 20. 1·40. 21. 1.
22. 2. 23. 2·70. 24. 0·80. 25. 5·30.

XXXIX. c. (p. 342).

1. 1·4150. 2. 3·4150. 3. 2·4232. 4. 3·4245. 5. 0·4245.
6. 2·4245. 7. $\bar{3}$·4245. 8. 412·4. 9. 2·428. 10. 24·28.
11. 242·8. 12. 0·0611. 13. 6955. 14. 57·82. 15. 5·558.
16. 798·4. 17. 8·002. 18. ·0634. 19. ·723. 20. ·002899.
21. 346700. 22. $8·371 \times 10^5$. 23. 4·5663. 24. 2·5663.
25. 3·1951. 26. 2·1951. 27. 2·6320. 28. 6·6320. 29. 3·3250.
30. 3·3250. 31. 3·5872. 32. $\bar{2}$·7568. 33. 6·8832. 34. 1·8836.
35. 2·448. 36. 2·646. 37. 1·913. 38. 1·58. 39. 1·47.
40. $x = 3·23$, $y = 1·52$. 41. 2·8, 10·55, 22·6, 3·8, 4·5. 42. 2·1505.
43. $\log 7$. 44. ·8. 46. 2921. 47. 1·46. 48. 14·04.
49. 0·1379. 50. 0·4651. 51. 191·4. 52. 0·0984. 53. 41·04.
54. 0·076. 55. 0·00052. 56. 0·0357. 57. 0·00231. 58. 13·57.
59. 0·9434. 60. 0·000016. 61. 1·334. 62. 54·88. 63. 41.
64. 3·74. 65. 7·14. 66. 0·8899. 67. 0·3183. 68. 0·098.

69. 11·47. 70. 0·298. 71. 15·47. 72. 560000. 73. 0·731.
74. 0·204. 75. 1·240, 1·433, 0·176, 0·30. 76. 31·65. 78. x.
80. 0·7927. 83. 5·82. 84. 0. 85. 0·2345. 86. 2·079.
87. (i) 105·5; (ii) 849·4. 88. 481·7; 7 lb. 89. 18·05 lb.
90. ·77; 2·43. 91. 4·8668967. 92. 3·8668996.
93. $\bar{2}$·8668990. 94. 736·034. 95. 7·36038.

XL. a. (p. 353).

1. 2·4 hrs. 2. 2·5 hrs. 3. $\frac{3}{5}$.
4. 6·7 days. 5. 3 days. 6. 4 a.m.
7. 4, 8, 12, 16, 20 min.; 352, 704, 1056, 1408 yds. from the start.
8. 12, 24, 36 min.; ·4, ·8, ·2 miles. 9. £27, 1330 marks. 10. 2·84.
11. 9 yds. 12. 7 miles an hour. 13. 2*s.* 10*d.* 14. 24 min.
15. 800 yds. 16. 2400 feet from start. 17. 11. 18. 7.
19. 30 miles. 20. 1 hr. 20 min. 21. 5·7 miles.
22. 5·14 miles. 23. $2\frac{3}{4}$ miles.
24. Ran 2 miles, distance 3 miles. Meet 21·2 min. and 60 min. from start, 1·4 and 1·5 miles from first man's starting point.
25. C travels 9·3 miles an hour, meets B 23·2 miles from first end.
26. 33·6 miles from B's start, in 4·8 hrs.
27. 5·25 hrs. 28. 1 hr.
29. (i) (1) 2.11. (2) 2.27. (3) 2.33.
(ii) (1) 6.33. (2) 6.16 and 6.49. (3) 6.11 and 6.55.
(iii) (1) 9.49. (2) 9.33. (3) 9.27.
(iv) (1) Never. (2) 11.42. (3) 11.33.
30. 4·05 days approx. 31. 8, 24 days.
32. 8, 12 gallons an hr. 33. 90, 180 days.
34. 70 %, 30 %. 35. 11 : 4.
36. 21·4 hrs. approx. and 50 hrs. 37. 2 hrs.
38. 9 miles an hour, 12 miles an hour.
39. 108000, 155520, 62500, 75000, 80000, 85000, 90000, 96000, 102000.
40. At 3.20, $13\frac{1}{3}$ miles from A. 41. 4 and 6 miles an hour.
42. 6 and 4 hours. 43. 45 miles; $1\frac{2}{3}$, $3\frac{1}{3}$ miles an hour.
44. 35 miles; 3·8 and 1·2 miles an hour. 45. $3\frac{3}{4}$, $2\frac{1}{2}$ hours.
46. 10, 4 hours. 47. A $2\frac{1}{2}$, B 2 miles an hour; dist. 5 miles.

XL. b. (p. 357).

1. 1020 yds. 2. 80. 3. 7200, 4800. 4. £1000. 5. 17, 4, or 20, 1.
6. 70 lb. 7. In 2 hrs. 32 min. after man leaves A, $8\frac{13}{15}$ miles from A.
8. £12, £10. 9. 22. 10. 9. 11. *bd* shillings. 12. 72.
13. 2. 14. 2 hrs. $55\frac{25}{77}$ min. 15. 1400.
16. $\frac{3}{5}$ pint from first, $\frac{2}{5}$ from second. 17. 10, 12 min.

18. $\frac{7}{34}$, $\frac{27}{34}$. **19.** 80. **20.** 1*s.* 4*d.*, 2*s.* 10*d.*
22. 13 innings, average 17. **23.** £5, £3. 17*s.* 6*d.* **24.** 20.
25. 159, 141. **26.** 6, 12, 36. **27.** 36 wheat, 42 barley. **28.** 39.

XLI. a. (p. 359).

1. $x^2 - 2x + 3 - 2x^{-1} + x^{-2}$; $a + 3a^{\frac{2}{3}}b^{\frac{1}{3}} + 3a^{\frac{1}{3}}b^{\frac{2}{3}} + b$. **2.** 3·01.
5. 54 : 35 ; 44 %. **6.** 17. **7.** $\frac{3}{2}$, -1, $-\frac{3}{2}$, 2.

XLI. b. (p. 359).

2. 1·442. **3.** 576 feet, 176 feet. **5.** $22\sqrt{2}$. **6.** 14. **7.** 10*d.*

XLI. c. (p. 360).

1. $\frac{3}{5}$. **2.** 5, 12, 13 in. **3.** $\frac{5}{3}(2\sqrt{6}+\sqrt{3})$. **4.** 4·65, $-0·65$.
6. Greater in the former case; (i) $\dfrac{\pi - 2}{\pi}$, (ii) $\dfrac{4-\pi}{4}$.

XLI. d. (p. 360).

1. $(ac + bd)^2 + (ad - bc)^2$, or $(ad + bc)^2 + (ac - bd)^2$.
2. 2, 4. **3.** ·097, 1·36, ·8.
4. 2 gals. from A, 14 from B. **6.** $\dfrac{45\sqrt{2}}{4}$. **7.** 2·596.

XLI. e. (p. 361).

1. $3 + \sqrt{5} = 5·236$. **2.** 12 miles per hour. **3.** ± 4.
5. 7, 24, 25 in. **7.** $1 \pm \sqrt{2} = 2·41$, $-·41$.

XLI. f. (p. 361).

1. 2, 18. **2.** £11. **3.** The latter.
5. $-\frac{1}{6} = -·1667$. **6.** £9. 10*s.* 9*d.* **7.** 1·707.

XLI. g. (p. 361).

1. 2. **2.** $\dfrac{1}{a}$ or a. **3.** ·4128. **5.** $w = 2x + 6z^2$.
6. (i) $x=4$ or 5, $y=5$ or 4. (ii) $x=3$.
7. at 5·17 miles from G, and 2·67 from G.

XLI. h. (p. 362).

1. $\{(ac+bd)e + (ad-bc)f\}^2 + \{(ac+bd)f - (ad-bc)e\}^2$. **2.** 50 feet.
3. -1, $-\frac{1}{2}$, -2. **6.** 1·8. **7.** 1·9452.

XLI. i. (p. 362).

1. $\frac{47}{2}\sqrt{5}$. **2.** $z = x - 3$ or $-2x + 5$. Factors $y - x + 3$ and $y + 2x - 5$.
3. 1·74. **4.** 1·27. **5.** $2\frac{1}{2}$ gallons. **7.** (i) $-1 \pm \sqrt{7}$. (ii) 0, $\pm\sqrt{3}$.

XLI. k. (p. 363).

1. 1·7 approx. 2. $\frac{a}{3}$. 3. 1920 yds.; $5\frac{1}{2}$ min.

4. $\frac{p^2-2q}{q}$. 5. $3+\sqrt{7}$. 6. 1. 7. $qy:px$.

XLI. l. (p. 363).

1 £48. 2. $\sqrt{2}$. 5. $\frac{3}{4}$ or $\frac{4}{3}$. 7. 2·327.

XLI. m. (p. 364).

2. $\frac{2\sqrt{5}}{3}(\sqrt{5}-1)$. 3. $3\frac{1}{3}$, $6\frac{2}{3}$. 4. 7.

5. $k=(a-1)(1-b)$. 6. $2:1$. 7. $32:25$.

XLI. n. (p. 364).

1. ·618. 3. 5, 6, 7 or 8. 4. 70. 5. 3. 6. £187. 10*s*.

7. $\frac{d}{t}\frac{nc}{(m-n)c-md}$, $\frac{d}{t}\cdot\frac{n(c-d)}{(m-n)c-md}$ yds. a minute;

distance$=\frac{(m-n)c(c-d)}{(m-n)c-md}$.

XLI. p. (p. 364).

1. $x=3$. 2. $5:12$. 3. 1·8733.

5. $2\frac{1}{4}$ feet. 6. 14 miles.

XLI. q. (p. 365).

1. $2x^2-\frac{3x}{2}+\frac{1}{2}$. 3. 14 miles. 4. $2:1$. 5. $1:3=5:\ldots$

6. ·1, ·316, ·562, 1·778, 3·162, 5·623; 1·585, 1·995, ·178.

7. $3^{\frac{1}{8}}=1{\cdot}147015\ldots$

XLII. (p. 368).

1. 7. 2. 40. 3. 23. 4. 70. 5. 24.
6. 4. 7. 7. 8. 8. 9. 26. 10. 45.
11. 8. 12. 20. 13. -7. 14. 47. 15. -34.
16. $3\frac{1}{2}$. 17. 320. 18. -44. 19. $2n-1$.
20. $l=14\frac{3}{4}$, $s=94\frac{1}{2}$. 21. $9\frac{1}{4}$, $61\frac{1}{2}$. 22. -115, -2075.
23. 5·25, 57·5. 24. $-2{\cdot}3$, $-24{\cdot}8$. 25. 0.
26. 17. 27. 133. 28. 163·2.
29. $n\{(5a-b)+2n(a-3b)\}$. 30. $15x+225$. 31. 75, 507.
32. 52, 297. 33. -27, -51. 34. -75, -438.
35. 5500. 36. $-1\frac{1}{2}$. 37. 48. 38. 6400.
39. $21\frac{7}{12}$. 40. $-169\frac{1}{2}$. 41. 2, 5, 8 …. 42. 46, 45, 44 ….

43. $-58, -51, -44 \ldots;$ **44.** $12\frac{3}{4}, 12\frac{1}{2}, 12\frac{1}{4} \ldots$.

45. $\frac{n}{2}(5n+13)$. **46.** $\frac{n}{2}(7n+13)$. **47.** 19. **48.** 7, 1390.

49. 17, 23, 29 .... **50.** $5\frac{2}{3}, 6\frac{1}{3}, 7 \ldots$. **51.** 32, 29, 26 ....

52. $x+\frac{2y}{3}, x+\frac{y}{3}, x \ldots$. **53.** 25. **54.** $\frac{5}{12}$.

55. 2940. **56.** 18, 348. **57.** 380. **58.** 2, 4, 6 ....

59. 8, 12, 16 .... **60.** 3 shillings.

61. In 207 days, 4140 miles west of the starting point.

62. 1560 yds., 1521 yds. **63.** In 17 hours; 68 miles from the start.

64. 90. **65.** 205. **66.** £7. 7*s*. 6*d*.

67. $a-2d, a-d, a, a+d, a+2d$. $a-\frac{3d}{2}, a-\frac{d}{2}, a+\frac{d}{2}, a+\frac{3d}{2}$.

68. $\frac{8a-9b}{6}, \frac{5a-3b}{3} \ldots$. **69.** $n=10$.

70. $3, 2\frac{1}{2}, 2 \ldots$. **71.** 1, 3, 5 ....

72. $\frac{1}{2}(2n+1)(a+b)$; $\frac{a+c}{2}$. **73.** $\frac{qm-pn+p-q}{m-n}$; $\frac{p-q}{m-n}$.

XLIII. (p. 373).

1. $3, 729, 3^n$. **2.** $3, 243, 3^{n-1}$. **3.** $3, \frac{3}{32}, \frac{3}{2^{n-1}}$.

4. $-\frac{1}{2}, -\frac{3}{32}, (-1)^{n-1}\frac{3}{2^{n-1}}$. **5.** $\frac{1}{4}, \frac{1}{2^{10}}, \frac{1}{2^{2n-2}}$. **6.** $-\frac{1}{4}, -\frac{1}{2^{10}}, \frac{(-1)^{n-1}}{2^{2n-2}}$.

7. $\frac{1}{3}, \frac{1}{27}, \frac{9}{3^{n-1}}=\frac{1}{3^{n-3}}$. **8.** $\frac{1}{2}, \frac{1}{4}, \frac{8}{2^{n-1}}=\frac{1}{2^{n-4}}$. **9.** $\frac{1}{x}, x^{n-6}, 1$.

10. $\frac{1}{x}, x^{n-2}, \frac{x^{n+3}}{x^{n-1}}=x^4$. **11.** $x, \frac{1}{x^{n-6}}, 1$. **12.** $-\frac{1}{x}, -\frac{1}{x^{n-6}}, (-1)^{n-1}$.

13. $2^{10}-1=1023$. **14.** $2-\frac{1}{2^7}=1\frac{127}{128}$. **15.** $4-\frac{1}{4^4}=3\frac{255}{256}$.

16. $\frac{1}{3}\left(2+\frac{1}{2^8}\right)=\frac{171}{256}$. **17.** $\frac{1}{3}\left(2-\frac{1}{2^7}\right)=\frac{85}{128}$. **18.** $\frac{a[1-(-x)^n]}{1+x}$.

19. $\frac{x^n-1}{x-1}$. **20.** $\frac{[x^n-(-1)^n]x}{x+1}$. **21.** 48. **22.** $121\frac{1}{2}$. **23.** $\frac{1}{256}$.

24. 3. **25.** $65535\frac{3}{4}$. **26.** $\frac{a(1-b^{2x})}{1-b^2}$. **27.** $\frac{1-c^{30}}{1+c^3}$.

28. $-\frac{31}{2}(2-\sqrt{2})$. **29.** $\frac{1}{2}$. **30.** $5\frac{2}{5}$. **31.** 16. **32.** $18\frac{2}{7}$.

33. $\frac{a^{\frac{3}{2}}}{b^{\frac{1}{2}}(a+b)}$. **34.** $\frac{40}{3}(3+\sqrt{3})$. **35.** $\frac{5^6-3^6}{2\times 5^4}=11\frac{573}{625}$. **36.** -1533.

38. $5\frac{58}{81}$. **39.** $4\frac{25}{27}$. **40.** $\frac{211}{81}(\sqrt{3}-\sqrt{2})$. **41.** $-a\frac{1-(-a)^n}{1+a}$.

42. 189. **43.** $21\frac{1}{2}$. **44.** $\pm 10, 20, \pm 40$. **45.** $\pm 27, \frac{27}{2}, \pm\frac{27}{4}$.

46. $\pm 14, 9\frac{1}{3}, \pm 6\frac{2}{9}$. **47.** $\pm 192, 384, \pm 768$. **48.** ± 21. **49.** ± 1.

50. ± 54. **51.** 1 or 0, according as n is odd or even. **52.** 32.

53. 127. **54.** 1. **55.** 83. **56.** -1. **57.** -25. **58.** $3^{n+1}-1$.
59. a^{n-6}. **60.** $\frac{1}{4}$. **61.** 4. **62.** 2^{n-3}. **63.** $a^{n+2}-b$.
65. $2(2^n-1)-n$. **66.** $\frac{8}{9}$. **67.** $\frac{68}{111}$. **68.** $\frac{61}{495}$. **69.** $\frac{199}{825}$.
72. ± 14, 28, ± 56. **73.** -6, 18. **74.** $\frac{1}{8}$.
76. $3+9+27+\ldots$ or $48-36+27\ldots$. **77.** The 7th. **79.** 76.
81. $\dfrac{x^{b-2c}(1-x^{nc})}{1-x^c}$. **82.** $2^{n+1}-3$.
83. $\dfrac{1}{2}\left(\dfrac{a}{1-r}\right)^2-\dfrac{a^2}{2(1-r^2)}$. **84.** $7\frac{1}{9}$ or $1\frac{1}{63}$.

XLIV. (p. 378).

1. $\frac{4}{11}, \frac{1}{3}, \frac{4}{13}$. **2.** $\frac{2}{9}, \frac{1}{4}, \frac{2}{7}$. **3.** $\frac{1}{6}, \frac{3}{23}, \frac{3}{28}$. **4.** -4.
5. $\frac{1}{39}$. **6.** $-\frac{3}{14}$. **7.** $1\frac{1}{5}, 1\frac{1}{2}$, 2, 3. **8.** $1\frac{19}{61}, 1\frac{19}{21}, 3\frac{11}{23}$.
9. $2\frac{2}{5}$, 3, 4, 6. **10.** $7, 5\frac{8}{11}, 4\frac{11}{13}$. **11.** $1\frac{1}{2}$. **16.** 9 or $\frac{1}{9}$.

XLV. a. (p. 383).

1. $\dfrac{n}{6}(2n^2+3n-5)$. **2.** $\dfrac{1-x^n}{(1-x)^2}-\dfrac{nx^n}{1-x}$. **3.** The 26th.
4. $\dfrac{a^3(r^{4n}-1)}{r^4-1}$. **5.** $\frac{1}{4}(n-1)n(n+1)(n+2)$. **6.** $n^2(n+1)$.
7. $\dfrac{n}{3}(n^2+6n+11)$. **8.** $\dfrac{n}{6}(n+1)(2an+a+3b)$. **9.** $\dfrac{n}{12}(2n^2+3n-1)$.
10. $a, 3a, 5a, 7a \ldots$. **11.** n^2, $\dfrac{1}{n(n+1)(n+2)}$. **12.** $\dfrac{5}{2}\left(1-\dfrac{2^n}{3^n}\right)$, $\dfrac{5}{2}$.
13. $\dfrac{36}{11}\left(1-\dfrac{5^{2n}}{6^{2n}}\right)$, $\dfrac{36}{11}$. **14.** $-2n$. **15.** $nx^2+n(n-1)x+\dfrac{n}{6}(n-1)(2n-1)$.
16. 2805. **17.** d. **19.** $\frac{4}{3}(2^{2n}-1)-n(n+1)$.
20. $n(n+1)$. **21.** $(2n-1)(2n+1)$. **22.** $n(n+1)(n+2)$.
23. $(a+2n-2)x^{n-1}$. **24.** $(2n-1)(2n+1)(2n+3)$. **25.** n^2+1.
26. $(-1)^{n-1}n(n+1)$. **27.** $(2n+3)(2n+5)$. **28.** $\dfrac{n}{2}(n+1)$, $\dfrac{n}{6}(n+1)(n+2)$.
29. $\dfrac{n}{3}(n+1)(n+2)$. **30.** $\dfrac{n}{3}(4n^2+6n-1)$. **31.** $\dfrac{n}{4}(n+1)(n+2)(n+3)$.
32. $\dfrac{a-(a+n-1)x^n}{1-x}+\dfrac{x(1-x^{n-1})}{(1-x)^2}$. **33.** $\dfrac{9}{4}-\dfrac{2n+3}{4\cdot 3^{n-1}}$.
34. $n(n+1)(2n^2+6n-1)-3n$. **35.** $\frac{2}{9}$.
36. $n^2(2n+1)$, $\frac{1}{6}n(n+1)(3n^2+5n+1)$. **37.** $(3n-2)(3n+1)$, $n(3n^2+3n-2)$.
38. (a) $\dfrac{1-(2n-1)x^n}{1-x}+\dfrac{2x(1-x^{n-1})}{(1-x)^2}$, (b) $\dfrac{1+x}{(1-x)^2}$. **39.** $4-\dfrac{n+2}{2^{n-1}}$.
40. 8. **41.** 0. **43.** (a) $\dfrac{n}{n+1}$, (b) 1.

44. (a) $\frac{n}{2n+1}$, (b) $\frac{1}{2}$. **45.** (a) $\frac{n(n+3)}{4(n+1)(n+2)}$, (b) $\frac{1}{4}$.

46. (a) $\frac{3}{4}-\frac{2n+3}{2(n+1)(n+2)}$, (b) $\frac{3}{4}$. **47.** $\frac{1}{a}\left(\frac{1}{x}-\frac{1}{x+na}\right)$.

48. $\frac{n(n+1)(n+2)(3n+5)}{12}$. **49.** $\frac{5}{4}-\frac{2n+5}{2(n+1)(n+2)}$. **50.** 120.

51. 220. **52.** 680. **53.** 285. **54.** 650.

55. 4. **56.** 224. **57.** 155. **58.** 174.

59. 28, 56. **60.** 6, 196. **61.** 30. **62.** 123,225.

XLVI. a. (p. 388).

1. $\frac{2}{3}$. **2.** a. **3.** 4, 11, 18. **4.** $\frac{1}{2}$. **6.** $\frac{n(n+1)}{2}a+nb$.

XLVI. b. (p. 388).

1. ·4142. **2.** 4 or 2. **4.** 2, 3, 6.

5. 13. **6.** $a^{\frac{1}{2}n\,.\,\overline{n+1}}$. **7.** 1·382, −4.

XLVI. c. (p. 388).

4. 3, 9, 15. **5.** a. **6.** $2(2^n-1)-\frac{n(n+1)}{2}$. **7.** 70.

XLVI. d. (p. 389).

1. 3, 7, 11, 15. **2.** $2\sqrt{b^2-c^2}$, $c^4x^2-4bc^2x+4b^2=0$. **3.** 20.

4. $9\frac{2}{3}$, $10\frac{1}{3}$; 460. **5.** 18. **6.** $\frac{1+e^2}{1-e^2}.h$.

XLVI. e. (p. 389).

1. $x^2+3px+2p^2+q=0$. **2.** A 16, B 40. **3.** 5·82.

4. $\frac{2}{3}n(n+1)(2n+1)$. **5.** $\frac{a}{4}, \frac{a}{16}, \frac{a}{64}\ldots$.

XLVI. f. (p. 390).

2. $2\sqrt{5}:\sqrt{13}$. **3.** $\frac{n}{3}(4n^2+6n-1)$. **4.** $\sqrt{17}$.

5. 4·919. **7.** 1·309 feet.

XLVI. g. (p. 390).

2. 60°, 108°. **4.** ·6309. **6.** 6 cm.

7. (2, 2). The line is a tangent there.

XLVI. h. (p. 390).

1. 3 and 4, 4 and 5, 7 and 8, 0 and 1, 0 and 1.

2. $5^{\frac{1}{5}}>6^{\frac{1}{6}}>7^{\frac{1}{7}}$; 2·46. **3.** 12. **4.** $3+2\frac{1}{2}+2+\ldots$.

5. 3·125 sq. in. **6.** 14, 3479. **7.** 5 secs.

XLVI. i. (p. 391).

2. $\dfrac{11}{18}-\dfrac{1}{3}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dfrac{1}{n+3}\right)$; $\dfrac{11}{18}$. **3.** 90, 110. **4.** diff. $=\dfrac{a}{3}+\dfrac{b}{2}$.
5. ± 4. **6.** 1, $\frac{1}{4}$.

XLVI. k. (p. 391).

1. 3. **2.** 14, 67. **4.** $\sqrt{2x-3}+\sqrt{x+2}$. **7.** $\dfrac{2}{n(n+1)}$, $\dfrac{2n}{n+1}$.

XLVI. l. (p. 392).

2. $\left\{\dfrac{n(n+1)}{2}\right\}^2-\left\{\dfrac{n(n-1)}{2}\right\}^2$. **3.** 1·242. **6.** 1595. **7.** $1\frac{1}{2}$ acres.

XLVI. m. (p. 392).

1. £4500, £2700. **2.** 38th; 10. **3.** $\frac{15}{4}$, 5, $\frac{25}{4}$.
4. $\dfrac{a}{8}(31+15\sqrt{2})$; $a=2(2-\sqrt{2})$. **5.** 4·983. **7.** $2\frac{1}{4}$ hrs.

XLVII. (p. 397).

1. 1155. **2.** 3050. **3.** 16215. **4.** 41t. **5.** 100e.
6. ·241. **7.** ·27065 .. **8.** ·54$\dot{1}$. **9.** 113·02. **10.** 678.
11. 1739. **12.** 2518. **13.** 1543. **14.** t968. **15.** t968.
16. 1360. **17.** 781. **18.** 4155. **19.** 100242.
20. 1t8e54. **21.** 222·22, 22·63$\dot{5}\dot{4}$. **22.** $\cdot\dot{3}$.
23 336532. **24.** 26·5. **25.** 30·25. **26.** 23784.
27. 2653919. **28.** 3tt·19. **29.** 1024; 605820, 345, 1756.
30. 423. **31.** 18·355. **32.** 6. **33.** 9. **34.** 5.
35. The denary number 17. **36.** Use the weights $1+2^2+2^4+2^6$.
37. Put in one scale 2 weights of 1 lb., and 1 each of 3, 3^2, 3^4; (2) put in one scale 3^4+3^3, in the other 3^2+3+1.
38. $3^5+3^4+3^2+1$ in one scale, 3^3+3 in the other.
39. Undetected errors, errors of 9 or a multiple of 9, errors of figures transposed, any error which does not alter the sum of the digits in the result or alters the sum by a multiple of 9.

XLVIII. a. (p. 404).

1. 6. **2.** 24. **3.** 6. **4.** 7. **5.** $n-6$. **6.** $n-r+1$.
7. $\dfrac{n}{n-r}$. **8.** 12; $^{18}P_3 - 816$. **9.** 840. **10.** 504.
11. 30240. **12.** 5040. **13.** 10. **14.** 6. **15.** 18. **17.** 7.
18. 84, 84. **19.** 56. **20.** 66. **21.** 210. **22.** 56. **23.** 286.
24. 1225, 60. **25.** 6. **26.** 7620. **27.** 9. **28.** 84.
29. 5. **30.** 9. **31.** 210, 120. **32.** 21. **33.** 35, 20.
34. 840. **35.** $\frac{1}{2}n(n-1)$. **36.** 60, 12. **37.** 96. **38.** 100.

XLVIII. b. (p. 410).

1. 6720. **2.** 1260. **3.** 1120. **4.** 720. **5.** 20. **6.** 840.
7. 24. **8.** 360. **9.** 120. **10.** $\underline{|n-1}$. **11.** 2880. **12.** 4.
13. 126. **14.** 7. **15.** 84. **16.** 7056. **17.** 1350.

XLVIII. c. (p. 414).

3. 10. **5.** 127. **6.** 255. **7.** 52. **8.** 28. **9.** 6 ; 2664.
10. 600. **11.** 78. **12.** 35. **13.** 252, 126, 126. **14.** 11760.
15. 4080. **16.** $\dfrac{\underline{|11}}{\underline{|5}\,\underline{|3}\,\underline{|2}}$ 27720. **17.** $\dfrac{\underline{|13}}{\underline{|3}\,\underline{|2}\,\underline{|2}\,\underline{|2}}$. **18.** 6.
19. $x=4$. **20.** $\underline{|8}=40320$, $3\,\underline{|7}=15120$. **21.** 24 ; 66660.
22. 9 ; 5005. **23.** 255. **24.** 502. **25.** ${}^7C_3\times\underline{|7}=176400$.
26. 80 ; 7. **27.** 60. **28.** $\dfrac{\underline{|12}}{\underline{|7}\,\underline{|5}}=792$. **29.** 3.
30. 495. **31.** 225. **32.** 35. **33.** 54.
34. 658. **35.** $\dfrac{\underline{|8}}{\underline{|3}\,\underline{|3}}=1120$, $\dfrac{\underline{|6}}{\underline{|3}\,\underline{|2}}=60$.

XLVIII. d. (p 415).

1. 1728. **2.** 24. **3.** ${}^{30}C_1+{}^{30}C_2+{}^{30}C_3+{}^{30}C_4+{}^{30}C_5$; ${}^{29}C_4$.
4. 3784 (*one person does not constitute a party*). **5.** 1440 ; 480.
6. 10. **7.** ${}^{11}C_8\times\underline{|10}\,\underline{|8}$. **8.** 35^2. **9.** 72.
10. $\dfrac{\underline{|52}}{24(\underline{|13})^4}$, $\dfrac{\underline{|52}}{(\underline{|13})^4}$. **11.** 15. **12.** $\dfrac{\underline{|15}}{6(\underline{|5})^3}$. **13.** $np+2$.
15. 18 in general, 15 when the two points lie on one of the lines.
16. 3360 ; 72 **18** $(n-2)(n-3)\,\underline{|n-2}$. **19.** $2^n-1-n-\frac{1}{2}n(n-1)$.
20. $\frac{1}{2}n(n-3)$. **21.** $\underline{|mn}/(\underline{|m})^n$. **23.** ${}^{m+1}C_n$
25. $\frac{1}{2}n(n-1)\cdot\frac{1}{2}m(m-1)+1$. **26.** $\frac{1}{4}mn(m-1)(n-1)$.

XLIX. a. (p. 421).

1. $n+1$. **2.** The 6th. **3.** a^{n-5}. **4.** $\dfrac{n(n-1)}{2}$.
5. $\dfrac{n(n-1)(n-2)(n-3)}{\underline{|4}}$. **6.** It contains x^r. **7.** $\underline{|r}$.
8. Put $-x$ for x, and so make alternate terms negative. **9.** The 9th.
10. 1, 4, 6, 4, 1 **11.** 1, 5, 10, 10, 5, 1. **12** 1, 6, 15, 20, 15, 6, 1.
13. $\dfrac{n-r+1}{r}x$. **14.** $\dfrac{n-r+1}{r}\cdot\dfrac{x}{a}$. **15.** The 4th.
16. 20. **17.** 256. **19.** 1st, 3rd, 5th, 7th.
20. 9th. **21.** 4th. **22.** 10. **23.** 10.

ANSWERS TO THE EXAMPLES.

PART II.

XXXV. a. (p. 284).

1. $x^5+ax^4+a^2x^3+a^3x^2+a^4x+a^5$.
2. $x^6-ax^5+a^2x^4-a^3x^3+a^4x^2-a^5x+a^6$.
3. $x^6+ax^5+a^2x^4+a^3x^3+a^4x^2+a^5x+a^6$.
4. $x^7-ax^6+a^2x^5-a^3x^4+a^4x^3-a^5x^2+a^6x-a^7$.
5. $x^7+ax^6+a^2x^5+a^3x^4+a^4x^3+a^5x^2+a^6x+a^7$.
6. $x^5+2x^4+4x^3+8x^2+16x+32$.
7. $x^5-2x^4+4x^3-8x^2+16x-32$.
8. $(x+1)(x-2)(x+2)$.
9. $(x^2+x-3)(x^2+x+1)$.
10. $\left(\frac{3}{x}-1\right)\left(\frac{9}{x^2}-\frac{3}{x}+1\right)$.
11. $\left(a+\frac{1}{a}\right)\left(a^2-1+\frac{1}{a^2}\right)$.
12. $(x-2)^2(x-1)(x-3)$.
13. $(x+1)(x-1)(y+1)(y-1)$.
14. $(c-a)(a-b+c)$.
15. $(x-1-\sqrt{3})(x-1+\sqrt{3})$.
16. $(a^2+5a+1)(a^2-5a+1)$.
17. $(ax-1)(bx^2+ax+1)$.
18. $(a-b)(a+b+c)$.
19. $(a+b+1)(a^2+b^2+1-a-b-ab)$.
20. $(x-5-\sqrt{2})(x-5+\sqrt{2})$.
21. $(2x-y-8)(2x-y+3)$.
22. $(x^2+2x+2)(x^2-2x+2)$.
23. $(x-2y-3)(x^2+y^2+9-6y+3x+2xy)$.
24. $(2x-2-\sqrt{3})(2x-2+\sqrt{3})$.
25. $(3x-4y+2)(3x+4y-5)$.
26. $(5x-2y+7)(3x-2y+4)$.
27. $(x^2+x+1)(x^2-x+1)$.
28. $(a-b+2)(a^2+b^2+4+ab-2a+2b)$.
29. $(4x+1-\sqrt{6})(4x+1+\sqrt{6})$.
30. $(4x^2+2x+1)(4x^2-2x+1)$.
31. $(2a+b-1)(4a^2+b^2+1-2ab+b+2a)$.
32. $(2x-1)(x-2)(2x^2-5x+5)$.
33. $(3x-5y+7)(2x-3y+3)$.
34. $(x^2+4x+8)(x^2-4x+8)$.
35. $(x^2+3xy+4y^2)(x^2-3xy+4y^2)$.
36. $(x+1)^2(x^2+2x-11)$.
37. $(x+a)(x^2+ax+a^2)(x^2-ax+a^2)$.
38. $(9a^2-3a+1)(9a^2+3a+1)$.
39. $(2x+3y-2)(x+y+4)$.
40. $(x^2+2x+4)(x^2-2x+4)$.
41. $(9x^2+12x+8)(9x^2-12x+8)$.
42. $(x-a)(x^2+ax+a^2)(x^2-ax+a^2)$.
43. $\left(x+4+\frac{1}{x}\right)\left(x-3+\frac{1}{x}\right)$.
44. $(x-1)(x-2)(x-3)$.
45. $\left(x+2+\frac{2}{x}\right)\left(x-7+\frac{2}{x}\right)$.
46. $(2x-1)(x+1)(x+2)$.
47. $\left(x+\frac{1}{x}\right)\left(x^2-6+\frac{1}{x^2}\right)$.
48. $\left(x-\frac{1}{x}\right)^2\left(x^2+5+\frac{1}{x^2}\right)$.

49. $x^2+4xy+y^2-9z^2$. **50.** $4x^2-9y^2+16z^2+16xz$.
51. $a^2+2ab+b^2-5a-5b+6$. **52.** $a^2+2ab+b^2-x^2-2xy-y^2$.
53. x^4+x^2+1. **54.** x^4+3x^2+4.
55. x^4-x^2-4x-4. **56.** $x^4+x^2y^2+25y^4$.
57. $4a^2+4ab+b^2-x^2-4xy-4y^2$. **58.** $4x^4-9x^2y^2+24xy^3-16y^4$.
59. $x^6+2x^5+x^4-x^2-2x-1$. **60.** x^4+4x^2+16.
61. $9(a^2-b^2)^2$. **62.** $x^2+1+\frac{1}{x^2}$.
63. $x^6-2x^4-7x^2-4$. **64.** $32x^6-32x^2$.
65. $6a^6x^2-6a^2x^6$. **66.** a^6-b^6.
67. $a^8-2a^4b^4+b^8$. **68.** $225x^4-244x^2+64$.
69. $36x^4-73x^2+16$. **70.** x^6-64.
71. $a^{12}-3a^8b^4+3a^4b^8-b^{12}$. **72.** $64a^8-128a^4b^4+64b^8$.
73. $a^4+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2$.
74. $5x^2+x+6$. **75.** $4x^2-1$.
77. $3x^2+7x+2$. **78.** $16-8b$.
80. x^2-b^2. **81.** $81a^4-72a^2b^2+16b^4$.
82. $x^3+x^2y+xy^2+y^3$. **83.** $x^3-x^2y+xy^2-y^3$.
84. $x^3+2x^2y+4xy^2+8y^3$. **85.** $x^3-2x^2y+4xy^2-8y^3$.

XXXV. b. (p. 292).

6. $3(a-b)(b-c)(c-a)$. **7.** $(a-b)(b-c)(c-a)(ab+bc+ca)$.
8. $-(x-y)(x+y)(y-z)(y+z)(z-x)(z+x)$.
9. $-(b-c)(c-a)(a-b)(a^2+b^2+c^2+bc+ca+ab)$. **19** 0.
20. $-\frac{1}{abc}$. **21.** 1. **22.** $(a-b)(b-c)(c-a)$. **23.** 0.
24. $\frac{1}{xyz}$. **25.** $\frac{a^2}{(a-x)(a-y)(a-z)}$. **26.** $\frac{3(b-c)(c-a)(a-b)}{(a+b)(b+c)(c+a)}$.
27. $ab+bc+ca$. **28.** 0. **29.** $\frac{1}{abc}$. **30.** 0.
31. $\frac{1}{a^2b^2c^2}$. **32.** 1. **33.** $-(a+b+c)$.
34. $\frac{1}{(x-a)(x-b)(x-c)}$. **35.** $\frac{x}{(x-a)(x-b)(x-c)}$.
36. $\frac{x^2}{(x+a)(x+b)(x+c)}$. **37.** -1. **38.** 0. **39.** 1.

XXXV. c. (p. 296).

1. $p=16a^2$. **4.** 4. **6.** $pq+r=0$.
11. $x=a$; $y=b$. **14.** $p^2+bp+ac=0$. **15.** $x=-b$, $y=a$.
16. 3. **17.** $8x+4$. **21.** $x=\frac{1}{2}$, $y=1$.
23. 3. **26.** $(2x^3-xy^2+3y)^2$. **27.** $1\frac{1}{3}$.
28. 0. **29.** $-(yz+zx+xy)$.

24. $p+1$. **25.** r. **26.** $r-1$. **27.** $r+2$.

28. (i) $(n-1)$th; (ii) $(n-3)$th; (iii) $(n-r+2)$th; (iv) $(n-r+1)$th.

29. (i) $(n-3)$th; (ii) $(n-6)$th; (iii) $(n-r+1)$th; (iv) $(n-r)$th.

30. (i) $(n+1)$th; (ii) $\frac{|\underline{2n}}{(|\underline{n})^2}a^n x^n$; (iii) $\frac{|\underline{2n}}{|\underline{n-1}\,|\underline{n+1}}a^{n-1}x^{n+1}$;

(iv) $\frac{|\underline{2n}}{|\underline{n+1}\,|\underline{n-1}}a^{n+1}x^{n-1}$.

31. ${}^nC_3 a^3 x^{n-3} + {}^nC_2 a^2 x^{n-2} + {}^nC_1 a x^{n-1} + x^n$.

32. (i) ${}^{25}C_{r-1} z^{r-1} x^{26-r}$; (ii) ${}^{25}C_r z^r x^{25-r}$; (iii) ${}^{25}C_{r-2} z^{r-2} x^{27-r}$;

(iv) ${}^{25}C_{r-1} z^{26-r} x^{r-1}$; (v) ${}^{25}C_r z^{25-r} x^r$.

XLIX. b. (p. 426).

1. $1+5x+10x^2+10x^3+5x^4+x^5$.

2. $a^7+7a^6x+21a^5x^2+35a^4x^3+35a^3x^4+21a^2x^5+7ax^6+x^7$.

3. $a^{12}-6a^{10}x^2+15a^8x^4-20a^6x^6+15a^4x^8-6a^2x^{10}+x^{12}$.

4. $a^8b^8 - 4a^7b^7x + 7a^6b^6x^2 - 7a^5b^5x^3 + \frac{35}{8}a^4b^4x^4 - \frac{7}{4}a^3b^3x^5 + \frac{7}{16}a^2b^2x^6 - \frac{1}{16}abx^7$
$+ \frac{1}{256}x^8$.

$\ldots x^6 + 28x^4 + 56x^2 + 70 + \frac{56}{x^2} + \frac{28}{x^4} + \frac{8}{x^6} + \frac{1}{x^8}$.

$\ldots a^6bx^5 + 21a^5b^2x^3 - 35a^4b^3x + \frac{35a^3b^4}{x} - \frac{21a^2b^5}{x^3} + \frac{7ab^6}{x^5} - \frac{b^7}{x^7}$.

$\ldots - 810x^4y + 1080x^3y^2 + 720x^2y^3 + 240x\ldots$

$\ldots 56a^6b^2 + 280a^4b^4 + 224\ldots$ **10.** $462a^{12}x^{10}$. **11.** $-1365\times 3^4y^{11}$.

12. 924. **13.** $-462\times 2^6x + 462\times\frac{2^5}{x}$.

14. $\frac{2n(2n-1)\dots(2n-r+1)}{|\underline{r}}x^r$; $(-1)^r\cdot\frac{n(n-1)\dots(n-r+1)}{|\underline{r}}\cdot 3^{n-r}2^r x^r$.

17. $\frac{n}{2}+1$. **18.** $\frac{|\underline{n}}{|\underline{r}\,|\underline{n-r}}2^r x^r$.

19. $(-1)^r\frac{|\underline{2n}}{|\underline{r}\,|\underline{2n-r}}\cdot 3^{2n-r}\cdot 5^r x^r$. **20.** $(-1)^r\frac{|\underline{n}}{|\underline{r}\,|\underline{n-r}}3^r a^{2n-2r}b^r x^r$.

21. $\frac{|\underline{n}}{\ldots|\underline{n-r}}x^{n-2r}$. **22.** $(-1)^n\frac{|\underline{2n}}{|\underline{n}\,|\underline{n}}$. **23.** $4032x^5y^4$, $2016x^4y^5$.

25. 21. **26.** 252×2^5.

$a^9 - 5a^8x + 8a^7x^2 - 14a^5x^4 + 14a^4x^5 - 8a^2x^7 + 5ax^8 - x^9$.

$x^8 - 2x^6 - 2x^4 + 6x^2 - \frac{6}{x^2} + \frac{2}{x^4} + \frac{2}{x^6} - \frac{1}{x^8}$. **30.** $1792\sqrt{3}$.

$-\frac{11x}{5}$. **36.** ${}^{14}C_7$. **37.** The 6th = the 7th = 7